On Coalgebras over Algebras

A. Balan[1,*]

Department of Mathematics II, University Politehnica of Bucharest, Romania

A. Kurz

Department of Computer Science, University of Leicester, UK

Abstract

We extend Barr's well-known characterization of the final coalgebra of a Set-endofunctor H as the completion of its initial algebra to the Eilenberg-Moore category $\mathsf{Alg}(\mathbf{M})$ of algebras associated to a Set-monad $\mathbf{M}$, if H can be lifted to $\mathsf{Alg}(\mathbf{M})$. As further analysis, we introduce the notion of commuting pair of endofunctors (T, H) with respect to a monad $\mathbf{M}$ and show that under reasonable assumptions, the final H-coalgebra can be obtained as the completion of the free $\mathbf{M}$-algebra on the initial T-algebra.

Keywords: Coalgebras, Algebras over a monad

---

1. Introduction

For any category $\mathcal{C}$ and any $\mathcal{C}$-endofunctor H, there is a canonical arrow between the least and the greatest fixed points of H, namely between its initial algebra and final coalgebra, assuming these exist. Functors for which these objects exist and coincide were called algebraically compact by Barr ([8])-for example, if the base category is enriched over complete metric spaces ([6]) or complete partial orders ([30]), then mild conditions ensure that the endofunctors are algebraically compact. However, if the category is just locally small, without any other enrichment, as Set, this coincidence does not happen. But there is still something to be said: Barr ([9]) showed that for bicontinuous Set-endofunctors, the final coalgebra can be realized as the completion of its initial algebra. This works whenever the functor does not map the empty set into itself, otherwise the initial algebra would be empty. Hence some well-known examples are lost, like functors obtained from powers and products. Barr's result was extended to all locally finitely presentable categories by Adámek

---

[*] Corresponding author

Email addresses: `asteleanu@yahoo.com` (A. Balan), `kurz@mcs.le.ac.uk` (A. Kurz)

[1] Supported by a Royal Society International Travel Grant and by the CNCSIS-UEFISCU project PN II-RU/2010 PD 56 no. 19/03.08.10.

([1], [2]), in the sense that the completion procedure works for hom-sets, not objects, with respect to all finitely presentable objects.

In the present paper we have focused on coalgebras whose carriers are algebras for a Set-monad, not necessarily finitary (see for example [11], [31]). Our interest arises from the following two developments. Firstly, streams or weighted automata, as studied by Rutten from a coalgebraic perspective ([26], [27], [28]) are mathematically highly interesting examples of coalgebras, despite the fact that the type functor is very simple, just $HX = \Bbbk \times X$ in the case of streams. The interesting structure arises from $\Bbbk$, which in typical examples is a semi-ring. In this paper, we shall bring this structure to the fore by lifting H to the category of modules for a semi-ring, or more generally, to the category of algebras for a suitable monad. Secondly, in recent work of Kissig and the second author ([19]), it turned out that it is of interest to move the trace-semantics of Hasuo-Jacobs-Sokolova ([13]) from the Kleisli-category of a commutative monad to the Eilenberg-Moore category of algebras (for example, this allows to consider wider classes of monads). Again, for trace semantics, semi-ring monads are of special interest.

In the first part of this paper, we show that Barr's theorem ([9]) extends from coalgebras on Set to coalgebras on the Eilenberg-Moore category of algebras $\mathsf{Alg}(\mathbf{M})$ for a monad $\mathbf{M}$ on Set, dropping the assumption $H0 \neq 0$ (hence allowing examples like the functor H of stream coalgebras mentioned above).

We consider the situation of a Set-endofunctor H that has a lifting to $\mathsf{Alg}(\mathbf{M})$. Under some reasonable assumptions, we are able to prove that the final H-coalgebra can be obtained as the Cauchy completion of the image of the initial algebra for the lifted functor, with respect to the usual ultrametric inherited from the final sequence. Moreover, the corresponding topology is compatible with the $\mathbf{M}$-algebra structure of both objects involved, in the sense that the algebra structure maps are continuous. To provide examples, we need to understand better the initial algebra of the lifted functor. This is the purpose of the second part of the paper, where the special case of an initial algebra that is free (as an $\mathbf{M}$-algebra) is exhibited. Namely, for two endofunctors H, T and a monad $\mathbf{M}$ on Set, we call (T, H) an $\mathbf{M}$-commuting pair if there is a natural isomorphism $HM \cong MT$, where M is the functor part of the monad. This notion is motivated by the fact that if both the algebra lift of H and the Kleisli lift of T exist, then mild requirements ensure that $\widetilde{H}$, the algebra lifted functor of H, is equivalent with the extension of T to $\mathsf{Alg}(\mathbf{M})$ if and only if they form a commuting pair (T, H). If this is the case, then one can recover the initial algebra for the lifted endofunctor $\widetilde{H}$ as the free $\mathbf{M}$-algebra built on the initial T-algebra. Consequently, the final $\widetilde{H}$-coalgebra can be realized as completion of a free $\mathbf{M}$-algebra.

An earlier version of this paper appeared as [7]. In the present article, Section 2.4 is extended with a detailed analysis of the completion result. Section 2.5, devoted only to examples, is new; in Appendix A, we start with a monad $\mathbf{M}$ and a functor H having a Kleisli lifting $\hat{H}$ and show how to extend $\hat{H}$ from free algebras to all algebras, in the form of a left Kan extension. Finally, due to space limitations, an example of constructing a commuting pair is detailed in Appendix B.

2. Final coalgebra for endofunctors lifted to categories of algebras

2.1. Final sequence for Set*-endofunctors*

Consider an endofunctor $H : \mathsf{Set} \longrightarrow \mathsf{Set}$. Denote by $\mathsf{Coalg}(H)$ the category of H-coalgebras and by $U_H : \mathsf{Coalg}(H) \longrightarrow \mathsf{Set}$ the corresponding forgetful functor. We are interested in the final object of $\mathsf{Coalg}(H)$, the final H-coalgebra.

Assumption I. *H preserves limits of ω^{op}-sequences.*

The above assumption ensures that the final H-coalgebra exists and can be obtained by the following well-known construction: from the unique arrow $t : H1 \longrightarrow 1$ we form the sequence

$$\begin{tikzcd} 1 & H1 \arrow[l, "t"'] & \cdots \arrow[l] & H^n 1 \arrow[l] & H^{n+1}1 \arrow[l, "H^n t"'] & \cdots \arrow[l] \end{tikzcd} \tag{1}$$

Denote by Z its limit, with $p_n : Z \longrightarrow H^n 1$ the corresponding cone. As we work in Set, recall that Z can be described as a subset of the cartesian product $\prod_{n\geq 0} H^n 1$, namely $Z = \{(x_n)_{n\geq 0} \mid H^n t(x_{n+1}) = x_n\}$. By applying H to the sequence and to the limit, we get a cone where additionally $HZ \longrightarrow 1$ is the unique map to the singleton set:

$$\begin{tikzcd} & & & HZ \arrow[llldd, bend right] \arrow[dd, bend left, "Hp_{n-1}"] & \\ & & & Z \arrow[u, dotted, "\cong"'] \arrow[llld, bend right] \arrow[d, "p_n"'] & \\ 1 & H1 \arrow[l, "t"'] & \cdots \arrow[l] & H^n 1 \arrow[l] & \cdots \arrow[l, "H^n t"'] \end{tikzcd}$$

The limit property and the assumption on H lead to a bijection $\xi : Z \longrightarrow HZ$ satisfying $Hp_{n-1} \circ \xi = p_n$ for all $n \geq 1$. We can now see that Z is the final H-coalgebra: for each H-coalgebra $(C, C \xrightarrow{\xi_C} HC)$, there is a cone $(C \xrightarrow{\alpha_n} H^n 1)_{n\geq 0}$ over the sequence (1), built inductively: $\alpha_0 : C \longrightarrow 1$ is the unique map; then given $\alpha_n : C \longrightarrow H^n 1$, construct α_{n+1} as the composite $C \xrightarrow{\xi_C} HC \xrightarrow{H\alpha_n} H^{n+1}1$. Then the unique map $\alpha_C : C \longrightarrow Z$ such that $p_n \circ \alpha_C = \alpha_n$ satisfies the following diagram:

$$\begin{tikzcd} C \arrow[r, "\alpha_C"] \arrow[d, "\xi_C"'] & Z \arrow[d, "\xi"] \\ HC \arrow[r, "H\alpha_C"] & HZ \end{tikzcd}$$

hence is a coalgebra map.

Now we have the final coalgebra. We move further and endow each set $H^n 1$ in (1) with the discrete topology (so all maps $H^n t$ will be continuous). Then put the initial topology ([29]) coming from this sequence on Z and HZ. It follows that ξ is a homeomorphism. In particular, the topology on Z is given by an ultrametric: the distance between any two points $x, y \in Z$ is 2^{-n}, for n the smallest natural number such that $p_n(x) \neq p_n(y)$. The cone $(C \xrightarrow{\alpha_n} H^n 1)_{n \geq 0}$ yields on any coalgebra a topology (the initial one) and the unique map $\alpha_C : C \longrightarrow Z$ is continuous with respect to it.

2.2. Lifting to the category of algebras for a monad

Let $\mathbf{M} = (M, M^2 \xrightarrow{m} M, Id \xrightarrow{u} M)$ be a monad on Set (by convention, we shall use a bold symbol for a monad, and an italic symbol for the underlying endofunctor). Denote by $\mathsf{Alg}(\mathbf{M})$ the Eilenberg-Moore category of $\mathbf{M}$-algebras and by $F^{\mathbf{M}} \dashv U^{\mathbf{M}} : \mathsf{Alg}(\mathbf{M}) \longrightarrow \mathsf{Set}$ the adjunction between the free and the forgetful functor. For later use, record that $\mathsf{Alg}(\mathbf{M})$ has an initial object, namely $(M0, M^2 0 \xrightarrow{m_0} M0)$, the free algebra on the empty set, and a terminal object 1, the singleton, with algebra structure given by the unique map $M1 \longrightarrow 1$.

Definition 2.1. *Let H be a Set-endofunctor. An algebra lifting of H is a functor $\widetilde{H} : \mathsf{Alg}(\mathbf{M}) \longrightarrow \mathsf{Alg}(\mathbf{M})$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{Alg}(\mathbf{M}) \arrow[r, "\widetilde{H}"] \arrow[d, "U^{\mathbf{M}}"'] & \mathsf{Alg}(\mathbf{M}) \arrow[d, "U^{\mathbf{M}}"'] \\
\mathsf{Set} \arrow[r, "H"] & \mathsf{Set}
\end{tikzcd} \tag{2}$$

Besides lifting H to algebras, our interest focuses on H-coalgebras. The next result seems to be "folklore"[2]:

Theorem 2.2. *The following are equivalent, for a monad $\mathbf{M}$ and an endofunctor H on a category $\mathcal{C}$:*

i. *Natural transformation $\lambda : MH \longrightarrow HM$ satisfying*

$$\begin{tikzcd}
H \arrow[r, "uH"] \arrow[dr, "Hu"'] & MH \arrow[d, "\lambda"] \\
 & HM
\end{tikzcd}
\qquad
\begin{tikzcd}
M^2H \arrow[r, "M\lambda"] \arrow[d, "mH"'] & MHM \arrow[r, "\lambda M"] & HM^2 \arrow[d, "Hm"] \\
MH \arrow[rr, "\lambda"] & & HM
\end{tikzcd} \tag{3}$$

ii. *Lifting of H to a functor $\widetilde{H}$ on $\mathsf{Alg}(\mathbf{M})$.*

---

[2] This is essentially a simplification of the case "monad, comonad and mixed distributive law" (also called an entwining) between them, as in [32], Thm.IV.1.

If this is the case, then the monad $\mathbf{M}$ *also lifts to a monad* $\widetilde{\mathbf{M}}$ *on* $\mathsf{Coalg}(H)$ *(a monad such that* $U_H\widetilde{M} = MU_H$*, see diagram below), the categories* $\mathsf{Coalg}(\widetilde{H})$ *and* $\mathsf{Alg}(\widetilde{\mathbf{M}})$ *are isomorphic and the adjunction* $F^{\mathbf{M}} \dashv U^{\mathbf{M}}$ *lifts to the adjunction* $\widetilde{F} \dashv \widetilde{U}$ *associated with the monad* $\widetilde{\mathbf{M}}$*, such that* $U_H\widetilde{U} = U^{\mathbf{M}}U_{\widetilde{H}}$ *and* $U_{\widetilde{H}}\widetilde{F} = F^{\mathbf{M}}U_H$*, as in the next diagram:*

$$\begin{tikzcd}
\mathsf{Coalg}(H) \arrow[loop above, "\widetilde{\mathbf{M}}"] \arrow[r, bend left, "\widetilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, "U_H"'] & \mathsf{Coalg}(\widetilde{H}) \cong \mathsf{Alg}(\widetilde{\mathbf{M}}) \arrow[l, bend left, "\widetilde{U}"] \arrow[d, "U_{\widetilde{H}}"] \\
\mathsf{Set} \arrow[loop below, "H"] \arrow[r, bend left, "F^{\mathbf{M}}"] \arrow[r, phantom, "\perp"] & \mathsf{Alg}(\mathbf{M}) \arrow[l, bend left, "U^{\mathbf{M}}"] \arrow[loop below, "\widetilde{H}"]
\end{tikzcd}$$

Proof. We just give an outline of the proof. The equivalence 1.$\Longleftrightarrow$ 2. is in [17]: for any $\mathbf{M}$-algebra (X, x), the functor $\widetilde{H}$ acts as HX, with algebra structure

$$MHX \xrightarrow{\lambda_X} HMX \xrightarrow{Hx} HX$$

and for any algebra map $(X, x) \longrightarrow (Y, y)$, the corresponding arrow $HX \longrightarrow HY$ respects the algebra structure. Conversely, given $\widetilde{H}$, one can recover the distributive law by defining first $\widetilde{\lambda} : F^{\mathbf{M}}H \longrightarrow \widetilde{H}F$ as the transpose of $H \xrightarrow{Hu} HM = HU^{\mathbf{M}}F^{\mathbf{M}} = U^{\mathbf{M}}\widetilde{H}F^{\mathbf{M}}$, then taking λ as

$$MH = U^{\mathbf{M}}F^{\mathbf{M}}H \xrightarrow{U\widetilde{\lambda}} U^{\mathbf{M}}\widetilde{H}F^{\mathbf{M}} = HU^{\mathbf{M}}F^{\mathbf{M}} = HM$$

Next, we can construct the monad $\widetilde{\mathbf{M}}$ as follows: on objects, it is

$$\widetilde{M}(C, C \xrightarrow{\xi_C} HC) = (MC, MC \xrightarrow{M\xi_C} MHC \xrightarrow{\lambda_C} HMC)$$

It has the same multiplication and unit as $\mathbf{M}$, but now restricted to H-coalgebras. It is easy to see that $\widetilde{H}$-coalgebras and $\widetilde{\mathbf{M}}$-algebras form isomorphic categories and the corresponding monadic adjunction $\widetilde{F} \dashv \widetilde{U}$ is explicitly given by: for any H-coalgebra $(C, C \xrightarrow{\xi_C} HC)$, $\widetilde{F}C = MC$ seen as free algebra, with coalgebra structure $MC \xrightarrow{M\xi_C} MHC \xrightarrow{\lambda_C} HMC$. Finally, $\widetilde{U}$ is the forgetful functor. $\square$

Remark 2.3. It is worth noticing that in general, the lifting is not necessarily unique (as there may be more than one distributive law $\lambda : MH \longrightarrow HM$). For example, take G a group and $HX = MX = G \times X$; consider H as an endofunctor and M as a monad with natural transformations u, m obtained from the group structure. The algebras for this monad are the G-sets. Then it is easy to see that a map $f : G \times G \longrightarrow G \times G$ induces a distributive law $\lambda : MH \longrightarrow HM$ if it satisfies $f(e, x) = (x, e)$ for all $x \in G$, where e stands for the unit of the group, and $f(\mu \times G) = (G \times \mu)(f \times G)(G \times f)$, where we have denoted by $\mu(x,y) = xy$ the group multiplication. Take now $f_1(x,y) = (xy, x)$ and $f_2(x,y) = (xyx^{-1}, x)$; these maps produce two distributive laws $\lambda_1, \lambda_2 : MH \longrightarrow HM$ that do not give same lifting $\widetilde{H}$, as the G-action on HX would be $(x,y,z) \longrightarrow (xy, x \rightharpoonup z)$ for λ_1, respectively $(x,y,z) \longrightarrow (xyx^{-1}, x \rightharpoonup z)$ for λ_2. Here $x, y \in G$, $z \in X$ and $\rightharpoonup$ denotes the left G-action on X. If the liftings were isomorphic, then the associated categories of coalgebras should also be isomorphic. In particular, notice that H is a comonad (as any set, in particular G, carries a natural comonoid structure) and both maps f_1, f_2 are actually inducing monad-comonad distributive laws λ_1, respectively λ_2. Hence each lifting carries a comonad structure, such that the associated categories of coalgebras for the lifted functors are Eilenberg-Moore categories of coalgebras and they should also be isomorphic. But for f_1, a corresponding coalgebra is the same as a G-set $(X, \rightharpoonup)$ endowed with a map $\theta : X \longrightarrow G$ such that $\theta(g \rightharpoonup x) = g\theta(x)$, while for the second structure, the compatibility relation yields a crossed G-set, i.e. $\theta(g \rightharpoonup x) = g\theta(x)g^{-1}$.

Assumption II. *There is a lifting of H to $\mathsf{Alg}(\mathbf{M})$, given by the distributive law $\lambda : MH \longrightarrow HM$.*

Remark 2.4. As the forgetful functor $\widetilde{U}$ creates and preserves limits, the existence of the final H-coalgebra $(Z, Z \xrightarrow{\xi} HZ)$ ensures that the final $\widetilde{H}$-coalgebra also exists, modeled on the same carrier. Hence despite the fact that the lifting might not be unique, the underlying set of the final H-coalgebra is preserved (but with possibly different algebra structures, depending on λ).

To see this, notice that any term $H^n 1$ of the final sequence (1) carries an $\mathbf{M}$-algebra structure, as follows: the obvious unique $\mathbf{M}$-algebra structure on 1, $a_0 : M1 \longrightarrow 1$; then, given $a_n : MH^n 1 \longrightarrow H^n 1$, define a_{n+1} as the composite $MH^{n+1}1 \xrightarrow{\lambda_{H^n 1}} HMH^n 1 \xrightarrow{Ha_n} H^{n+1}1$.

All maps in the sequence (1) are easily proved to be $\mathbf{M}$-algebra maps using the commutative diagrams in (3). Consequently, there is a unique $\mathbf{M}$-algebra structure $\gamma : MZ \longrightarrow Z$ such that the second row in the diagram below is limiting in $\mathsf{Alg}(\mathbf{M})$, in particular the projections $p_n : Z \longrightarrow H^n 1$ are $\mathbf{M}$-algebra morphisms:

$$\begin{tikzcd}
M1 \arrow[d, "a_0"'] & MH1 \arrow[l, "Mt"'] \arrow[d, "a_1"'] & \cdots \arrow[l] & MH^n1 \arrow[l] \arrow[d, "a_n"'] & MH^{n+1}1 \arrow[l, "MH^nt"'] \arrow[d, "a_{n+1}"'] & \cdots \arrow[l] & MZ \arrow[lll, bend right, "Mp_n"'] \arrow[d, dotted, "\gamma"] \\
1 & H1 \arrow[l, "t"'] & \cdots \arrow[l] & H^n1 \arrow[l] & H^{n+1}1 \arrow[l, "H^nt"'] & \cdots \arrow[l] & Z \arrow[lll, bend left, "p_n"]
\end{tikzcd}$$

As $U^{\mathbf{M}}$ creates limits and H is ω^{op}-continuous (Assumption I), so is $\widetilde{H}$. Therefore $\xi : Z \longrightarrow HZ$ is an isomorphism in $\mathsf{Alg}(\mathbf{M})$:

$$\begin{tikzcd}
MZ \arrow[r, "M\xi"] \arrow[d, "\gamma"'] & MHZ \arrow[r, "\lambda_Z"] & HMZ \arrow[d, "H\gamma"] \\
Z \arrow[rr, "\xi", "\cong"'] & & HZ
\end{tikzcd} \tag{4}$$

So $((Z,\gamma),\xi)$ is the final $\widetilde{H}$-coalgebra. Additionally, the cone $(MZ \xrightarrow{Mp_n} MH^n1 \xrightarrow{a_n} H^n1)_{n\geq 0}$ coincides with the cone $(\alpha_n : MZ \longrightarrow H^n1)_{n\geq 0}$ induced by the H-coalgebra structure of MZ from (4), as $a_n \circ Mp_n = p_n \circ \gamma$ and γ is the unique coalgebra map which makes (4) commute.

2.3. Topology on the final coalgebra

Remember that on all H^n1 we have considered the discrete topology. Endow also all MH^n1 with the discrete topology (intuitively, this corresponds to the fact that operations on algebras with discrete topology are automatically continuous) and MZ with the initial topology coming from the cone $(MZ \xrightarrow{Mp_n} MH^n1)_{n\geq 0}$.

Proposition 2.5. *The final H-coalgebra is a topological $\mathbf{M}$-algebra*[3]*, i.e. the $\mathbf{M}$-algebra structure map on Z, $\gamma : MZ \longrightarrow Z$, is continuous with respect to the topologies on Z and MZ.*

Proof. By definition of the initial topology, γ is continuous if and only if all compositions $p_n \circ \gamma$ are continuous. But $p_n \circ \gamma = a_n \circ Mp_n$, a_n are continuous as maps between discrete sets and Mp_n are continuous by the initial topology on MZ. ☐

The above proposition can be interpreted by saying that all operations on Z are continuous (as they are obtained as limits of operations on discrete algebras). This result relies heavily on the construction of the final coalgebra as the limit of the sequence (1), i.e. on the property of H to preserve limits of ω^{op}-chains. We do not know at this moment if Proposition 2.5 still holds if one drops this assumption, as there is no obvious choice for the topology on MZ. However, there is a possible direction to follow: instead an ω^{op}-continuous endofunctor, to use a finitary one. Following Worrell's construction ([33]), the final sequence would still induce a topology on Z, and the easiest way would be to take on MZ the initial topology with respect to γ, but this is not the same as the construction pursued here.

---

[3] Usually the notion of a topological algebra refers to an algebra for some finitary, algebraic theory whose underlying set is equipped with some topology, such that the algebra operations are continuous ([18]). As Eilenberg-Moore algebras for a Set-monad are the same as algebras for (not necessarily) finitary algebraic theories ([3]), we find that the term "topological algebra" characterizes best the present situation.

2.4. Initial $\tilde{H}$-algebra and final $\tilde{H}$-coalgebra in Alg($\mathbf{M}$)

If $\widetilde{H}$ preserves colimits of ω-sequences, then the initial $\widetilde{H}$-algebra is easy to build, using a dual procedure to the one in (1): recall that $\mathsf{Alg}(\mathbf{M})$ has an initial object, namely the free algebra on the empty set, $F^{\mathbf{M}}0 = (M0, M^2 0 \xrightarrow{m_0} M0)$. In order to simplify the notation, we shall identify all algebras $\widetilde{H}^n F^{\mathbf{M}}0$ with their underlying sets $H^n M0$. Then it is well-known that the initial $\widetilde{H}$-algebra is the colimit in $\mathsf{Alg}(\mathbf{M})$ of the chain

$$M0 \xrightarrow{!} HM0 \xrightarrow{H!} \ldots \longrightarrow H^n M0 \xrightarrow{H^n!} \ldots \tag{5}$$

where $! : M0 \longrightarrow HM0$ is the unique algebra map. Denote by $i_n : H^n M0 \longrightarrow I$ the colimiting cocone. We do not detail anymore this construction as we did for coalgebras as it will not be used in the sequel. However, we shall need the following (which requires only the existence in $\mathsf{Alg}(\mathbf{M})$ of the limit of the final sequence (1), respectively of the colimit of the initial sequence (5)): there is a unique $\mathbf{M}$-algebra morphism $f : I \longrightarrow Z$ such that

$$\begin{tikzcd}
H^n M0 \arrow[r, "i_n"] \arrow[d, "H^n s"'] & I \arrow[d, "f"] \\
H^n 1 & Z \arrow[l, "p_n"]
\end{tikzcd} \tag{6}$$

commutes for all n (see for example [1], Lemma 2.4), where $s : M0 \longrightarrow 1$ is the unique algebra map from the initial to the final object in $\mathsf{Alg}(\mathbf{M})$. If $M0$ is not empty, then I will also be not empty, as it comes with a cocone of algebra maps with not empty domains.

We shall generalize in this section Barr's result ([9]) from Set to $\mathsf{Alg}(\mathbf{M})$, for the special case of $\mathsf{Alg}(\mathbf{M})$-endofunctors arising as liftings of Set-endofunctors. The proofs use similar ideas to the ones in [9] and [1].

We start by assuming the existence of an algebra map $1 \longrightarrow M0$. By initiality of $M0$ and finality of 1, this implies $M0 \cong 1$, hence we have a zero object in the category of algebras.

Remark 2.6. There is a large class of Set-monads satisfying this condition. Some examples are: the list (or word) monad $MX = X^*$, the multiset monad $MX = \{f : X \longrightarrow \mathbb{N} \mid \mathsf{supp}(f) < \infty\}$, the power-set monad $MX = \mathcal{P}X$, the lift monad $MX = 1 + X$ or the sub-distribution monad $MX = \{f : X \longrightarrow [0,1] \mid \mathsf{supp}(f) < \infty, \sum_{x \in X} f(x) \leq 1\}$. For all these, the free algebra with empty set of generators is built on the singleton set. But there are also monads for which the carrier of the free algebra on the empty set has more than one element, as the exceptions monad $MX = E + X$ (with E a set with more than one element) and the double contravariant power set monad monad $MX = \mathcal{P}(\mathcal{P}(X))$, or it is empty, as is the case for the monad $MX = X \times \mathfrak{M}$, for $\mathfrak{M}$ a monoid. It is still under work whether the results of the present paper still hold without the assumption $M0 = 1$. For more details, we send the reader to [21], [22], [13]. We add for later use that all the mentioned monads with $M0 = 1$ are commutative, except the list monad.

We have $! : 1 = M0 \longrightarrow HM0 = H1$ and $t \circ ! = Id$ in $\mathsf{Alg}(\mathbf{M})$. Hence in the final sequence (1) all morphisms are split algebra maps, the colimit is the initial $\widetilde{H}$-algebra and the limit is the final H (and $\widetilde{H}$)-coalgebra:

$$1 \underset{!}{\overset{t}{\leftrightarrows}} H1 \leftrightarrows ... \leftrightarrows H^n 1 \underset{H^n !}{\overset{H^n t}{\leftrightarrows}} H^{n+1}1 \leftrightarrows ... \tag{7}$$

Theorem 2.7. *Let H be an ω^{op}-continuous Set-endofunctor and $\mathbf{M}$ a monad on Set such that:*

- i. *$M0 = 1$;*
- ii. *H admits a lifting $\widetilde{H}$ to $\mathsf{Alg}(\mathbf{M})$;*
- iii. *The lifted functor $\widetilde{H}$ is ω-cocontinuous.*

Then the carrier of the final H-coalgebra is the Cauchy completion of the image of the initial $\widetilde{H}$-algebra under a suitable (ultra)metric and this completion is compatible with the algebra structure (in the sense that both objects involved become topological algebras).

Proof. Consider the following diagram (in $\mathsf{Alg}(\mathbf{M})$), where all algebras involved have structure maps defined via the distributive law λ, as explained immediately after Remark 2.4:

$$\begin{tikzcd}
1 \arrow[r, shift left, "!"] & H1 \arrow[l, shift left, "t"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & H^n 1 \arrow[l, shift left] \arrow[r, shift left, "H^n !"] \arrow[dl, "i_n"'] & \cdots \arrow[l, shift left, "H^n t"] \\
 & & I \arrow[rr, "f"'] & & Z \arrow[ul, "p_n"]
\end{tikzcd}$$

Put on I the smallest topology such that f is continuous, where Z has the structure of a topological algebra from Proposition 2.5. This coincides with the initial topology given by the cone $I \xrightarrow{f} Z \xrightarrow{p_n} H^n 1$. Then I becomes a topological algebra if on MI we take the topology induced by the map $Mf : MI \longrightarrow MZ$. In particular, Mf is continuous.

Denote by $MI \xrightarrow{\zeta} I$ the algebra structure map of I. Then $f \circ \zeta = \gamma \circ Mf$ (remember that f is an algebra map). As Z is a topological $\mathbf{M}$-algebra, it follows that $f \circ \zeta$ is continuous, hence ζ is continuous.

About $(i_n)_{n \geq 0}$: these maps are by construction algebra morphisms, being the components of the colimiting cocone in $\mathsf{Alg}(\mathbf{M})$, and also continuous, as all $H^n 1$ are discrete.

It remains to be proved that I (more precisely, the image of I under f) is dense in Z.

First, use that limits in $\mathsf{Alg}(\mathbf{M})$ are computed as in Set to conclude that Z is Cauchy complete under the ultrametric defined in Section 2.1: take a Cauchy sequence $(x^{(n)})_{n \geq 0}$ in Z with respect to the initial topology (ultrametric) and assume $d(x^{(n)}, x^{(m)}) < 2^{-\min(m,n)}$ for all m, n. This implies $p_n \circ f(x^{(n)}) = p_n \circ f(x^{(m)})$ for all $n < m$. Thus $y = (p_n \circ f(x^{(n)}))_{n \geq 0}$ defines an element of Z and $\lim x^{(n)} = y$.

Next, we have to show that the image of I under the algebra morphism f is dense in Z. For this purpose, consider the additional $\mathbf{M}$-algebra sequence of morphisms $(h_n)_{n\geq 0}$, given by

$$h_n : Z \xrightarrow{p_n} H^n 1 = H^n M0 \xrightarrow{H^n !} H^{n+1}M0 \xrightarrow{i_{n+1}} I \xrightarrow{f} Z$$

We have $p_{n+1} \circ h_n = H^n! \circ p_n$. Consider now an element $x \in Z$. Then by construction $(y^{(n)} = h_n(x))_{n\geq 0}$ form a sequence of elements lying in the image of f and we shall see that this sequence is convergent to x. Indeed, from $p_{n+1}(y^{(n)}) = H^n! \circ p_n(x)$ it follows that

$$p_n(y^{(n)}) = H^n t \circ p_{n+1}(y^{(n)}) = H^n t \circ H^n! \circ p_n(x) = p_n(x)$$

As the n-th projection of the n-th term of the sequence $(y^{(n)})_{n\geq 0}$ coincides with the n-th projection of the element x, we have $d(y^{(n)}, x) < 2^{-n}$, implying $\lim y^{(n)} = x$ in Z. Therefore the image of I through the canonical colimit-limit arrow is dense in Z. ☐

Remark 2.8. If we consider on the initial algebra I the final topology coming from the ω-chain, this is exactly the discrete topology (and metric), since all $H^n 1$ are discrete, hence I would be Cauchy complete and $f : I \to Z$ automatically continuous. No interesting connection between I and Z can be obtained in this situation.

Remark 2.9. The idea of equipping the limit of an ω^{op}-sequence in Set with an ultrametric (obtained by considering each component of the sequence to be discrete) goes back to Barr ([9]) and has also been applied by Adámek in [2]. In the first quoted paper, the sequence in discussion is the final sequence of a Set-functor and the limit is its final coalgebra; in [2], the sequence is derived from the final sequence of an endofunctor on a locally finitely presentable category, by applying the functor $\mathsf{hom}(B, -)$, for each locally presentable object B. Although we apply the same construction of the ultrametric, what we have new is the compatibility between the topological structure and the algebra structure, on both initial $\widetilde{H}$-algebra I and final $\widetilde{H}$-coalgebra Z.

In Theorem 2.7, the proof of the completeness of the final coalgebra uses a similar argument to the one in [9]. For the density of the initial algebra, the construction of the sequence $(y^{(n)})_{n\geq 0}$ is borrowed from [2].

We look now at the last condition in Theorem 2.7, which requires the ω-cocontinuity of the lifted endofunctor. This happens, for example, if the functor H itself and the forgetful functor $U^{\mathbf{M}}$ from algebras to sets preserve colimits of ω-chains: consider a chain of algebras $X_0 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$. Then $HU^{\mathbf{M}}(\text{colim } X_n) \cong H(\text{colim } U^{\mathbf{M}}X_n) \cong \text{colim } HU^{\mathbf{M}}X_n$ by the above assumption; but $HU^{\mathbf{M}} = U^{\mathbf{M}}\widetilde{H}$ and $U^{\mathbf{M}}$ reflects isomorphisms, hence the canonical map $\text{colim } \widetilde{H}X_n \longrightarrow \widetilde{H}(\text{colim } X_n)$ is an isomorphism; in particular, if H and M are ω-cocontinuous, then $\widetilde{H}$ will also be.

In the case the monad is finitary there is more to say: one can drop the assumption on the cocontinuity of $\widetilde{H}$, necessary only to ensure the convergence (in ω steps) of the initial $\widetilde{H}$-sequence. To see this, start by noticing that $\widetilde{H}$ preserves monomorphisms: by the finitarity of $\mathbf{M}$, the forgetful functor $U^{\mathbf{M}}$ preserves monomorphisms ([10]), no $\mathbf{M}$-algebra is empty (if $M0 = 1$ is assumed) and H preserve injective maps with nonempty domains (as any Set-functor). Then we have:

Proposition 2.10. *Let H be an ω^{op}-continuous Set-functor and $\mathbf{M}$ a finitary monad such that $M0 = 1$ and a lifting of H to $\mathsf{Alg}(\mathbf{M})$ exists. Then the initial algebra of the lifted endofunctor exists and it is a subobject of its final coalgebra.*

Proof. As H is continuous, the lifted functor $\widetilde{H}$ will also be so. We have just seen that it preserves monomorphisms. Now the result follows from [2], Prop. 3.4. $\square$

We can now rephrase Theorem 2.7 as follows:

Theorem 2.11. *Let H be a Set-endofunctor that preserves limits of ω^{op}-chains and $\mathbf{M}$ a finitary monad on Set such that:*

i. *H admits a lifting $\widetilde{H}$ to $\mathsf{Alg}(\mathbf{M})$;*
ii. *$M0 = 1$ in $\mathsf{Alg}(\mathbf{M})$.*

Then the final H-coalgebra is the Cauchy completion of the initial $\widetilde{H}$-algebra under a suitable (ultra)metric.

Proof. By the previous proposition, the initial algebra for $\widetilde{H}$ exists and can be computed in $\mathsf{Alg}(\mathbf{M})$ as the colimit of the initial sequence, possibly after more than ω steps (see [2], 2.2 for the transfinite construction of the initial sequence). Notice that all we needed in the proof of Theorem 2.7 was the existence of algebra maps $i_n : H^n 1 \longrightarrow I$ and $f : I \longrightarrow Z$ such that $p_n \circ f \circ i_n = Id$. Such morphisms are easily seen to exist also in the present situation, thus the proof of Theorem 2.7 applies and the desired result follows. $\square$

Before ending this Section, we want to make a connection between Theorem 2.7 and Adámek's results ([2]). Assume that the monad $\mathbf{M}$ is finitary. Recall that in this case the category $\mathsf{Alg}(\mathbf{M})$ is locally finitely presentable, with finitely presentable objects the reflexive coequalizers of free algebras on finite sets ([12]). In [2], Thm. Sect. 3, the set of algebra maps $\mathsf{Alg}(\mathbf{M})(B, I)$ is shown to be dense in $\mathsf{Alg}(\mathbf{M})(B, Z)$, for every finitely presentable algebra B, where $\mathsf{Alg}(\mathbf{M})(B, Z)$ has the limit topology obtained by applying $\mathsf{Alg}(\mathbf{M})(B, -)$ to the final sequence (1) and considering each hom-space endowed with the discrete topology. In the present situation, take first B to be a free algebra $B = M(\underline{m})$ with $\underline{m}$ finite set of m elements. Then we can identify the hom-algebra spaces with finite powers, as $\mathsf{Alg}(\mathbf{M})(B, -) = \mathsf{Alg}(\mathbf{M})(M(\underline{m}), -) \cong \mathsf{Set}(\underline{m}, -) \cong (-)^m$; in particular, as each $H^n 1$ is a discrete algebra, its finite power $(H^n 1)^m$ will also be discrete (algebra). The resulting topology on $\mathsf{Alg}(\mathbf{M})(M(\underline{m}), Z) \cong Z^m$ will then coincide with the product topology coming from Z; in particular the completion result of [2] (applied for $B = M(\underline{m})$) follows, as all spaces involved are finite products of H^n1, I and Z respectively. Move now to any finitely presentable algebra B and write it as a quotient $M(\underline{m}) \longrightarrow B$. Then we obtain the diagram

$$\begin{tikzcd}
\cdots \arrow[r, shift left] & \mathsf{Alg}(M(\underline{m}), H^n1) \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] & \mathsf{Alg}(M(\underline{m}), I) \arrow[r] & \mathsf{Alg}(M(\underline{m}), Z) \arrow[lll, bend right] \\
\cdots \arrow[r, shift left] & \mathsf{Alg}(B, H^n1) \arrow[l, shift left] \arrow[r, shift left] \arrow[u] & \cdots \arrow[l, shift left] & \mathsf{Alg}(B, I) \arrow[r] \arrow[u] & \mathsf{Alg}(B, Z) \arrow[u] \arrow[lll, bend left]
\end{tikzcd}$$

in which each vertical arrow is injective. Endow each $\mathsf{Alg}(M(\underline{m}), H^n1)$ with the discrete topology; then the topologies on $\mathsf{Alg}(M(\underline{m}), I)$ and $\mathsf{Alg}(M(\underline{m}), Z)$ have been described above. The initial topology on $\mathsf{Alg}(B, Z)$ induced from the discrete sequence $(\mathsf{Alg}(B, H^n1))_{n\geq 0}$ can be easily seen to be the same as the topology induced from $\mathsf{Alg}(M(\underline{m}), Z)$; similarly, the topology on $\mathsf{Alg}(B, I)$ induced from $\mathsf{Alg}(B, Z)$ coincides with the one from $\mathsf{Alg}(M(\underline{m}), I)$. Consequently, the Cauchy completion result in [2] follows from the density of I in Z.

2.5. Examples

A. Consider a Set-endofunctor H. When does a monad satisfying conditions i. and ii. of Theorem 2.7 exist? We notice that the answer is always positive, if H is not the constant functor mapping everything to the empty set. For then $H1 \neq 0$; assuming the axiom of choice, there is (at least) one map $\alpha : 1 \longrightarrow H1$. Then an example of a Set-monad satisfying conditions i. and ii. of Theorem 2.7 is the lift monad $MX = 1 + X$. The distributive law $1 + HX \longrightarrow H(1 + X)$ is the cotupling of $1 \xrightarrow{\alpha} H1 \xrightarrow{H(inl)} H(1 + X)$ and $HX \xrightarrow{H(inr)} H(1 + X)$, where $inl : 1 \longrightarrow 1 + X$, $inr : X \longrightarrow 1 + X$ are the canonical injections into the coproduct. Later, in Example 2.13.i., we shall see a lifting obtained in this simple situation.

B. Instead, one could start with a monad and look for endofunctors H that lift to algebras. Then:

- The identity functor trivially lifts to algebras for any monad.
- The functor part of the monad has a lifting, by the distributive law $M^2 \xrightarrow{m} M \xrightarrow{Mu} M^2$. The lifted functor will be $F^{\mathbf{M}}U^{\mathbf{M}}$ (the functor part of the comonad associated with the adjunction $F^{\mathbf{M}} \dashv U^{\mathbf{M}}$).
- If H is a constant functor at some set $\Bbbk$, then liftings of H to the category $\mathsf{Alg}(\mathbf{M})$ are in one-to-one correspondence with $\mathbf{M}$-algebra structures on $\Bbbk$.
- If $HX = X \times \Bbbk$, and $\Bbbk$ carries an $\mathbf{M}$-algebra structure, then it is easy to see that a lift of H exists, as the forgetful functor $U^{\mathbf{M}}$ preserve products. Conversely, if $\widetilde{H}$ is a lifting of H, then there is an algebra structure on $\Bbbk$, namely $\widetilde{H}1$.

- if $HX = X^A$, a power functor, then the lifting exists as the forgetful functor $U^{\mathbf{M}}$ preserves limits.
- In fact, the last two situations exposed come from the following observation ([15]): given an I-indexed collection of functors $H_i : \mathsf{Set} \longrightarrow \mathsf{Set}$, $i \in I$, with distributive laws $\lambda_i : MH_i \longrightarrow H_iM$, there is a distributive law $\lambda : M\prod\limits_{i\in I} H_i \overset{[M\pi_i]}{\longrightarrow} \prod\limits_{i\in I} MH_i \overset{\prod\limits_{i\in I}\lambda_i}{\longrightarrow} \prod\limits_{i\in I} H_iM$, where π_i stands for the canonical projection from the product to the i-th component.
- if $HX = A + X$ or $HX = X + X$, there is no obvious distributive law $\lambda : MH \longrightarrow HM$, unless the monad preserves coproducts (which is the case, for example, if the monad has a right adjoint, like $MX = X \times \mathfrak{M}$, for $\mathfrak{M}$ a monoid). If this is the case, then functors like $H = \coprod_{i\in I} H_i$, where each H_i has a distributive law $\lambda_i : MH_i \longrightarrow H_iM$, lift to Eilenberg-Moore algebras by $M\coprod\limits_{i\in I} H_i \cong \coprod\limits_{i\in I} MH_i \overset{\coprod\limits_{i\in I}\lambda_i}{\longrightarrow} \coprod\limits_{i\in I} H_iM$.

Concluding the above, a class of (polynomial) functors for which a lifting to Eilenberg-Moore algebras exists for an arbitrary Set-monad (no conditions at all on the monad) can be obtained from the identity functor, constant functors to sets that are carriers of $\mathbf{M}$-algebras, products with such sets and arbitrary powers. Additionally, one can consider also functors as MG, for any G, with distributive law $M^2G \overset{m_G}{\longrightarrow} MG \overset{MGu}{\longrightarrow} MGM$ (which also cover the second case above, for $G = Id$).

Assumption III. *From now, we shall consider that all conditions in Theorem 2.7 are fulfilled.*

Regarding the behavior of the endofunctor H on the empty set, there are two possible situations: $H0 = 0$ or not. We shall discuss each of them in the next paragraphs.

C. If $H0 = 0$, the initial H-algebra will be empty. However, when lifted to algebras, the situation changes (this was our original motivation) as the initial $\widetilde{H}$-algebra is no longer empty (remember that $M0$ is not empty).

Consider as a reference example the functor $HX = X^A \times \mathbb{k}$, built from products and powers, hence ω-continuous. The H-coalgebras are known as deterministic automata with input set A and ouputs in $\mathbb{k}$: the transition map of a coalgebra $X \longrightarrow X^A \times \mathbb{k}$ sends a state $x \in X$ to a pair: the associated next-state function $next(x) : A \longrightarrow X$ that works when receiving an input from A, and an output $out(x) \in \mathbb{k}$. The final H-coalgebra is $\mathbb{k}^{A^*}$, the set of all behaviour functions that map a finite sequence of inputs to the last observable output in $\mathbb{k}$ (here $A^* = \coprod_{n\geq 0} A^n$ denotes the free monoid of finite words on A). Such a functor always admits at least one lifting to $\mathsf{Alg}(\mathbf{M})$ for any monad $\mathbf{M}$, provided $\mathbb{k}$ is a structured output set, i.e. it carries an algebra structure (see paragraph

B). The lifted functor is given by the same formula as H, where this time the product and the power are computed in the category of algebras.

There are some particular cases of this functor that are worth mentioning:

Example 2.12. We start with the easiest case, when $A = 1$; then $A^* = \mathbb{N}$. The coalgebras for the functor $HX = X \times \Bbbk$ are usually called stream automata; the final coalgebra, denoted $\Bbbk^\omega$, consists of infinite sequences of consecutively observed output values. Particularize for $\Bbbk = \mathbb{R}$, the case of real-valued stream coalgebras as in [27]. Consider now the monad that sends a set X to the vector space with basis X. Explicitely, $MX = \{\sum_{i \in I} r_i x_i \mid r_i \in \mathbb{R}, x_i \in X, I \text{ finite set}\}$ is the set of formal finite linear combinations of elements of X with real coefficients. The functor $HX = X \times \mathbb{R}$ lifts to the category of vector spaces by the distributive law $M(X \times \mathbb{R}) \longrightarrow MX \times \mathbb{R}$, saying that scalar multiplication is performed component-wise (a finite linear combination of pairs with real coefficients $\sum_i r_i(x_i, s_i)$ is mapped to the pair $(\sum_i r_i x_i, \sum_i r_i s_i)$ formed by a formal linear combination and a real number). The lifted functor $\widetilde{H}$ sends a vector space X into the direct sum $X \oplus \mathbb{R}$. From any coalgebra $(C, C \xrightarrow{\xi_C} HC)$, we obtain an $\widetilde{H}$-coalgebra by allowing linear combinations of transitions. Then the sequence (7) becomes

$$(0) \rightleftarrows \mathbb{R} \rightleftarrows \cdots \rightleftarrows \mathbb{R}^n \rightleftarrows \mathbb{R}^{n+1} \rightleftarrows \cdots$$

with the natural embeddings $(r_1, \ldots, r_n) \mapsto (r_1, \ldots, r_n, 0)$ and projections $(r_1, \ldots, r_n, r_{n+1}) \mapsto (r_1, \ldots, r_n)$ as component maps. The limit of the above sequence, the final coalgebra of real streams $\mathbb{R}^\omega$, is a vector space with addition and scalar multiplication defined component-wise. The colimit of the above is the subspace of $\mathbb{R}^\omega$ of all streams with finitely many components nonzero. Each $\sigma \in \mathbb{R}^\omega$ is the limit of a sequence $(\sigma_n)_{n \geq 0}$ formed with such streams, namely $\sigma_n = (\sigma(0), \sigma(1), \ldots, \sigma(n), 0, 0, \ldots)$.

Example 2.13. We take now arbitrary set of inputs A, but $\Bbbk = \{0, 1\}$. The transition function of a coalgebra $X \longrightarrow HX = X^A \times \{0, 1\}$ provides binary outputs, deciding if a state is accepting (response 1) or not. The final sequence (1) has components $H^n 1 \cong \mathcal{P}(A_n)$, where A_n denotes the set of words of length less than n over A. For any n, the connecting map $H^{n+1} 1 \longrightarrow H^n 1$ restricts a language $L \in \mathcal{P}(A_{n+1})$ to $L \backslash A^n$, the sub-language formed only of words of length less than n. The final coalgebra $\{0, 1\}^{A^*}$ can be identified with the set of formal languages $\mathcal{P}(A^*)$ over A. There are several monads for which $\Bbbk$ carries an algebra structure (hence allowing a lift); here we mention only two of them.

i. First, one can consider $\Bbbk$ as a pointed set, i.e. an algebra for the lift monad $MX = 1 + X$. This can be achieved in two ways, depending which distinguished element is chosen, leading thus to different liftings. Notice that $H1 \cong \{0, 1\}$, thus the choice of the distinguished element is equivalent with the choice of the map $\alpha : 1 \longrightarrow H1$ from paragraph A. An easy calculation can show that the lifting constructed as explained in paragraph A. is the same as the one discussed at the beginning of paragraph C.

(i.1) We start with the case $\Bbbk = (\{0,1\}, 0)$. Denote by $\widetilde{H}_0$ the lifted functor. Then $\widetilde{H}_0$ will send a pointed set $(X, x \in X)$ to the set $HX = X^A \times \{0,1\}$ with distinguished element $(f_x, 0)$, where f_x designates the constant function from A to X, mapping all inputs in A to x. Practically, this means that an automaton is extended by adding one state which does not accept anything. The lifted sequence (7) will have components $\widetilde{H}_0^n 1 = (\mathcal{P}(A_n), \emptyset)$, with connecting maps (notice the preservation of distinguished elements):

- $\widetilde{H}_0^n 1 \longrightarrow \widetilde{H}_0^{n+1} 1$ is the inclusion: a language $L \in \mathcal{P}(A_n)$ formed of words of length less than n can be also seen as a subset of A_{n+1};
- $\widetilde{H}_0^{n+1} 1 \longrightarrow \widetilde{H}_0^n 1$ is the restriction, as described earlier, to sub-languages of words of length less than n.

The limit of the sequence (the final coalgebra) $\mathcal{P}(A^*)$ is again a pointed set in $\emptyset$, with projections $p_n : \mathcal{P}(A^*) \longrightarrow \mathcal{P}(A_{n-1})$, $p_n(L) = L \backslash (A^n A^*)$[4], sending a language to the sub-language formed of words of length less than n.

The ultrametric inherited from the sequence: the distance between two languages L and L' is 2^{-n} iff L and L' share same words of length less than n, but differ on length n.

The colimit of the (increasing) sequence is easily seen to be $\bigcup_{n \geq 0} \mathcal{P}(A^n)$, the set of all bounded languages on A (in the sense that the length of words in the language is bounded), again with the empty set as pointed structure.

The completion works as follows: take any language $L \in \mathcal{P}(A^*)$ and define L_n as the subset of L containing all words in L of length less than n; then $(L_n)_{n \geq 0}$ is a sequence of bounded languages converging to L. If A is finite, which is the usual assumption on deterministic automata, then the colimit is just the set of all finite languages.

(i.2) Consider now $\Bbbk = (\{0,1\}, 1)$. Although the underlying sets in the sequence (7) are the same as earlier, it is instructive to see how the arrows and the structure of the pointed sets change: the lifted functor $\widetilde{H}_1$ sends now (X, x) to $(X^A \times \{0,1\}, (f_x, 1))$. It means that H-coalgebras are enriched with one more state, which accepts everything. The components of the final sequence are $\widetilde{H}_1^n 1 = (\mathcal{P}(A_n), A_n)$, with connecting maps:

- $\widetilde{H}_1^n 1 \longrightarrow \widetilde{H}_1^{n+1} 1$ sends a language $L \in \mathcal{P}(A_n)$ to $L + A^n$ (adds all words of length n);
- $\widetilde{H}_0^{n+1} 1 \longrightarrow \widetilde{H}_0^n 1$ acts the same as before, restricting a language $L \in \mathcal{P}(A_{n+1})$ to $L \backslash A^n$, the sub-language formed only of words of length less than n.

---

[4] Here $A^n A^*$ denotes the set of words of length at least n.

The topology on the final coalgebra is the same as above, but the distinguished element changes to A^*.

The initial $\widetilde{H}_1$-algebra will be $I = \bigcup_{n\geq 1}\{L + A^n A^* \mid L \in \mathcal{P}(A_n)\}$, with A^* as distinguished point, and maps $i_n : (\mathcal{P}(A_n), A_n) \longrightarrow I$, $i_n(L) = L + A^n A^*$. We might call I the set of all *co-bounded* languages over A (although we do not know if this terminology is used), in contrast to the previous case. The density of I in $\mathcal{P}(A^*)$ is obtained in the following way: for a language $L \in \mathcal{P}(A^*)$, consider in I the sequence $(L_n + A^n A^*)_{n>0}$, with L_n defined as above, the subset of all words in L of length less than n. Then for each $n > 0$, the distance between L and $L_n + A^n A^*$ is at most 2^{-n}, hence this sequence converges to L. Again, in the case A finite, we recognize the colimit as the pointed set of cofinite languages.

ii. The second monad we are interested in is the power-set monad. The $\mathcal{P}$-algebras are sup-lattices (posets having all suprema); morphisms of $\mathcal{P}$-algebras are the sup-preserving maps. Consider $\Bbbk = \{0, 1\}$ as a sup-lattice with the usual order $0 < 1$. The lifted endofunctor $\widetilde{H}$ will send a sup-lattice X to the product of sup-lattices $X^A \times \{0, 1\}$, where X^A inherits the order from X. The components of the sequence (7) are again $\widetilde{H}^n 1 = \mathcal{P}(A_{n-1})$, but this time seen with the free sup-lattice structure, with sup-preserving maps:

   - $\widetilde{H}^n 1 \longrightarrow \widetilde{H}^{n+1} 1$ sends a language $L \in \mathcal{P}(A_{n-1})$ to itself, seen now as a subset of A_n;
   - $\widetilde{H}^{n+1} 1 \longrightarrow \widetilde{H}^n 1$ restricts a language $L \in \mathcal{P}(A_n)$ to the sub-language $L \backslash A^n$ formed only of words of length less than n.

   We obtain thus a sequence of embedding-projections pairs, hence the limit $\mathcal{P}(A^*)$ is also the colimit ([30]), with canonical maps $(i_n)_{n\geq 0}$ left adjoint to the projections $(p_n)_{n\geq 0}$. In this situation, the realization of the final H-coalgebra as a completion of the initial $\widetilde{H}$-algebra is trivial, as the latter is as large as possible.

Example 2.14. Consider A a finite set and $\Bbbk$ a semi-ring (not necessarily commutative). Recall that a semi-ring is a set equipped with two operations: addition and multiplication, and two constants, denoted 0 and 1, such that $(\Bbbk, +, 0)$ is a commutative monoid and $(\Bbbk, \cdot, 1)$ is a monoid. The two structures are connected by the usual distributive laws ([27]). For a semi-ring $\Bbbk$, the construct $MX = \{f : X \longrightarrow \Bbbk \mid \mathrm{supp}(f) \text{ finite}\}$, where $\mathrm{supp}(f) = \{x \in X \mid f(x) \neq 0\}$, induces a monad (as in [21], Section VI.4, Ex. 2, where the ring R is replaced by the semi-ring $\Bbbk$). Algebras for this monad are called $\Bbbk$-modules: they are commutative monoids, with an external operation of multiplication with elements of $\Bbbk$; in particular, $M0$ is the zero module, with trivial operations. Coming back to the functor $HX = X^A \times \Bbbk$, its final H-coalgebra can be identified with the formal power series in non-commuting A variables, while the initial $\widetilde{H}$-algebra is the direct sum of A^* copies of $\Bbbk$ (the polynomial algebra in the same variables) (in this case, finite products and coproducts coincide in $\mathsf{Alg}(\mathbf{M})$).

The approximants of order n in the corresponding ω-sequence are $H^n 1 = \Bbbk^{1+A+\ldots+A^n}$, the polynomials in (non-commuting) A-variables of degree at most n. We shall provide details of this for the easiest case, where A is the singleton $\{t\}$; the distance between two elements of the final coalgebra $\Bbbk[[t]]$, i.e. between two power series $f(t), g(t)$ in variable t, is given precisely by $2^{-ord(f(t)-g(t))}$, where $ord(f(t) - g(t))$ is the order of the difference $f(t) - g(t)$ (the smallest power of t that occurs with a nonzero coefficient in the difference). Take a Cauchy sequence of polynomials $f_n(t) = a_0^n + a_1^n t + \ldots$, where only finitely many a_j^n are nonzero, for each $n, j \in \mathbb{N}$. For every $r \geq 0$, there is an n_r such that for every $n \geq n_r$, we have $ord(f_n(t) - f_{n_r}(t)) = r$; this implies $a_j^n = a_j^{n_r}$ for all $j \leq r$ and $n \geq n_r$. Let $f(t) = a_0^{n_0} + a_1^{n_1} t + \ldots$. One immediately verifies that the power series $f(t)$ is the limit of of the sequence $(f_n(t))_{n \geq 0}$. Hence the final coalgebra $\Bbbk[[t]]$ is indeed the completion of the initial $\widetilde{H}$-algebra $\Bbbk[t]$.

D. Suppose now that $H0 \neq 0$. In addition, we shall require that H preserves colimits of ω-sequences. Then the initial H-algebra J exists and is the Set-colimit of the sequence $0 \longrightarrow H0 \longrightarrow \ldots \longrightarrow H^n 0 \longrightarrow \ldots$. Correspondingly, in the diagram below, the first row and the second row have a colimit, respectively a limit in Set, while the colimit of the second row is computed in $\mathsf{Alg}(\mathbf{M})$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H0 \arrow[r] \arrow[d] & \cdots \arrow[r] & H^n 0 \arrow[r] \arrow[d] \arrow[rr, bend left] & \cdots & J \arrow[d, "g"'] \arrow[dr, bend left, "h"] & \\
1 \arrow[r, shift left] & H1 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & H^n 1 \arrow[l, shift left] \arrow[r, shift left] \arrow[rr, bend left] & \cdots \arrow[l, shift left] & I \arrow[r, "f"] \arrow[ll, bend left] & Z
\end{tikzcd}$$

There is a unique map $J \xrightarrow{g} I$ induced from the cocone $H^n 0 \longrightarrow H^n 1 \longrightarrow I$; if h is the initial algebra-final coalgebra arrow for H in Set, then $h = f \circ g$, with f given by (6). As h is injective by Barr's theorem, g will also be injective. The topology chosen for Z is the same as in Section 2.1. Because both topologies on J and I are induced by the one on Z, g is also continuous, hence I should provide a better approximation of the final coalgebra (see next example). However, if $H0 = 1$ and the forgetful functor $U^{\mathbf{M}}$ preserves colimits of ω-chains (which is the case if the monad does), it follows that the carrier of the initial $\widetilde{H}$-algebra I coincides with the initial H-algebra J.

Example 2.15. Consider the the functor $HX = A + X \times X$. H-coalgebras are binary systems with exceptions in A (like termination, deadlock, etc.). The initial H-algebra is known to be the set of all finite binary trees with leaves labeled in A, while the final coalgebra contains the finite and infinite trees. Assume A is a non-empty semigroup (for example, any nonempty set A, with binary operation $ab = a$, where $a, b \in A$). For the list monad $MX = X^*$, a distributive law $\lambda_X : (A + X \times X)^* \longrightarrow A + X^* \times X^*$ can be described as follows: if a word $w \in (A + X \times X)^*$ contains at least one entry from A, then $\lambda_X(w)$ will be the product in A of all entries in w belonging to A, in the order they appear, forgetting thus all other entries from $X \times X$ (for the example mentioned above, this reduces to the first element of A in w). If not, then $w = [(x_1, y_1), \ldots, (x_n, y_n)]$ with $x_i, y_i \in X$ and take $\lambda_X(w) = ([x_1, \ldots, x_n], [y_1, \ldots, y_n])$. In particular, $\lambda_X(\epsilon) = (\epsilon, \epsilon)$, where ϵ denotes the empty word. Now, remember that the $\mathbf{M}$-algebras are exactly the monoids. The lifted functor $\widetilde{H}$ sends a monoid $(X, \cdot, e_X)$ to $A + X \times X$, with component-wise multiplication on $X \times X$, while the multiplication on A comes from its semigroup structure; in addition, elements of A are absorbing. The unit element will be (e_X, e_X). The components of the initial $\widetilde{H}$-sequence: for $n \geq 0$, $H^n M0$ is the set of all finite binary trees of depth at most n with leaves labeled in $A + \{*\}$, modulo the following equivalence relation on trees: if a node has both children leaves with label $*$, then this node is considered itself a leaf, again labeled in $\{*\}$. For the monoid structure, notice first that multiplying two trees with only one node (labeled in A) produces a tree again with one node, labeled in the A-product of labels, and second, that these trees with one node A-labeled are absorbing with respect to trees of greater depth. Next, the multiplication on trees is defined inductively: if two trees t_1 and t_2 have children t_{11} and t_{12}, respectively t_{21} and t_{22}, then $t_1 \cdot t_2$ is the tree whose root has children $t_{11} \cdot t_{21}$ and $t_{12} \cdot t_{22}$. The tree with only one node labeled in $\{*\}$ is the unit element. The arrows are the inclusions. It follows that the colimit I is the monoid of all finite binary trees with leaves labeled in $A + 1$, modulo the same equivalence relation as above and with the same monoid structure. We now describe the completion procedure: consider a tree t in the final coalgebra Z. If t is finite, then it belongs also to I, hence the constant sequence on t will give the result. If not, define t_n to be the n-th cutting of t, with label $*$ to all terminal nodes which had children in t. Then the sequence $(t_n)_{n \geq 0}$ belongs to I and it is convergent to t, with respect to the ultrametric described in [2], 3.8.(a).

3. M-commuting pairs of endofunctors

In the previous section we have considered a Set-endofunctor H which admits an algebra lifting $\widetilde{H}$ with respect to a monad $\mathbf{M}$. We have seen that under some assumptions, the carrier of the final H-coalgebra (which coincides with the carrier of the final $\widetilde{H}$-coalgebra) can be obtained as the Cauchy completion of the carrier of the initial $\widetilde{H}$-algebra. We shall discuss in this section under which conditions the initial $\widetilde{H}$-algebra is free as an $\mathbf{M}$-algebra and can be realized by a "similar" construction (in the sense of extending a functor from Set to $\mathsf{Alg}(\mathbf{M})$, see below).

Recall that there are two ways of relating an endofunctor H on Set (or on any other category) to a monad $\mathbf{M}$, using a natural transformation, as follows:

- $\lambda : MH \longrightarrow HM$ satisfying (3), which is the same as an algebra lift $\widetilde{H} : \mathsf{Alg}(\mathbf{M}) \longrightarrow \mathsf{Alg}(\mathbf{M})$, $U^{\mathbf{M}}\widetilde{H} = HU^{\mathbf{M}}$;

- $\varsigma : HM \longrightarrow MH$ satisfying

$$\begin{tikzcd}
H \arrow[r, "Hu"] \arrow[rd, "u_H"'] & HM \arrow[d, "\varsigma"] & HM^2 \arrow[r, "\varsigma_M"] \arrow[d, "Hm"'] & MHM \arrow[r, "M\varsigma"] & M^2H \arrow[d, "m_H"] \\
& MH & HM \arrow[rr, "\varsigma"] & & MH
\end{tikzcd} \tag{8}$$

It is well known that this is equivalent to the existence of a Kleisli lift, i.e. an endofunctor $\hat{H} : Kl(\mathbf{M}) \longrightarrow Kl(\mathbf{M})$ such that $\hat{H}F_{\mathbf{M}} = F_{\mathbf{M}}H$, where $F_{\mathbf{M}} : \mathsf{Set} \longrightarrow Kl(\mathbf{M})$ is the canonical functor to the Kleisli category of the monad. In this case, we can perform an additional construction: denote by $\mathcal{I} : Kl(\mathbf{M}) \longrightarrow \mathsf{Alg}(\mathbf{M})$ the comparison functor. Take the $\mathsf{Alg}(\mathbf{M})$-endofunctor $\bar{H}$ given by the left Kan extension along $\mathcal{I}$ (for the construction of $\bar{H}$, see Appendix A, for $\mathcal{C} = \mathsf{Set}$).[5] Composing the natural isomorphism $\mathcal{I}\hat{H} \cong \bar{H}\mathcal{I}$ with $F_{\mathbf{M}}$, we obtain $\bar{H}F^{\mathbf{M}} \cong F^{\mathbf{M}}H$, as in the diagram below:

$$\begin{tikzcd}
\mathsf{Alg}(\mathbf{M}) \arrow[rrr, dotted, bend left, "\bar{H}"] & Kl(\mathbf{M}) \arrow[l, "\mathcal{I}"'] \arrow[r, "\hat{H}"] & Kl(\mathbf{M}) \arrow[r, "\mathcal{I}"] & \mathsf{Alg}(\mathbf{M}) \\
& \mathsf{Set} \arrow[ul, "F^{\mathbf{M}}"] \arrow[u, "F_{\mathbf{M}}"'] \arrow[r, "H"] & \mathsf{Set} \arrow[u, "F_{\mathbf{M}}"] \arrow[ur, "F^{\mathbf{M}}"'] &
\end{tikzcd}$$

We shall call $\bar{H}$ an extension of H to algebras.

With the above notations, consider now two Set-functors T, H such that both an algebra lift of H and a Kleisli lift of T exist and $\widetilde{H} \cong \bar{T}$. Then we have

$$\begin{aligned} MT &= U^{\mathbf{M}}F^{\mathbf{M}}T \cong U^{\mathbf{M}}\bar{T}F^{\mathbf{M}} \\ &\cong U^{\mathbf{M}}\widetilde{H}F^{\mathbf{M}} = HU^{\mathbf{M}}F^{\mathbf{M}} = HM \end{aligned}$$

i.e. M acts like a switch (up to isomorphism) between the endofunctors T and H.

Definition 3.1. *Let $\mathbf{M} = (M, m, u)$ be a monad on* Set. *A pair of* Set-*endofunctors (T, H) such that $HM \cong MT$ is called an* $\mathbf{M}$-*commuting pair.*

Example 3.2. One can easily obtain commuting pairs in the following situations:

- Take $T = H = Id$ or $T = H = M$ and $\mathbf{M} = (M, m, u)$ any monad;
- Consider $T = H = A + (-)$, $\mathbf{M} = B + (-)$. Then commutativity of the coproduct ensures the commuting pair; similarly for products: $T = H = A \times (-)$, $\mathbf{M} = B \times (-)$, where this time B is a monoid (this works more generally, in any monoidal category).

---

[5] For any Set-monad $\mathbf{M}$, the category $\mathsf{Alg}(\mathbf{M})$ has coequalizers ([4]).

To the best of our knowledge, it seems that the notion of commuting pair has not been considered previously, although the above examples show that it arises naturally in mathematics. We shall later see more (non-trivial) examples.

Assumption IV. *Through the rest of the paper, we shall consider H, T finitary* Set*-endofunctors and* $\mathbf{M}$ *a finitary* Set*-monad.*

Proposition 3.3. *Assume that H has an algebra lift $\widetilde{H}$ and T has a Kleisli lift with respect to the monad $\mathbf{M}$. Denote by $\bar{T}$ the corresponding left Kan extension. Then:*

- i. *If $\widetilde{H} \cong \bar{T}$, then (T, H) form an $\mathbf{M}$-commuting pair and for any set X, the corresponding natural bijection $HMX \cong MTX$ is an isomorphism of $\mathbf{M}$-algebras, where the algebra structure of HMX is induced by the distributive law $\lambda : MH \longrightarrow HM$ and MTX is seen as a free $\mathbf{M}$-algebra.*[6]
- ii. *Conversely, if (T, H) form an $\mathbf{M}$-commuting pair such that the natural bijection $MTX \cong HMX$ is an algebra morphism (with $\mathbf{M}$-algebra structures on HMX and MTX as before), then $\widetilde{H} \cong \bar{T}$.*

Proof. 1. If $\widetilde{H} \cong \bar{T}$, then $\widetilde{H}F^{\mathbf{M}} \cong \bar{T}F^{\mathbf{M}} \cong F^{\mathbf{M}}T$, which can be rephrased by saying that $HMX \cong MTX$ is an isomorphism of $\mathbf{M}$-algebras, i.e. the following diagram commutes:

$$\begin{tikzcd}
MHMX \arrow[r, "\lambda_{MX}"'] \arrow[d, "\cong"'] & HM^2X \arrow[r, "Hm_X"'] & HMX \arrow[d, "\cong"] \\
M^2TX \arrow[rr, "m_{TX}"] & & MTX
\end{tikzcd}$$

where the right vertical arrow comes from $HM \cong MT$, while the left arrow is obtained by applying M to it.

2. From $HM \cong MT$ and

$$HM = HU^{\mathbf{M}}F^{\mathbf{M}} = U^{\mathbf{M}}\widetilde{H}F^{\mathbf{M}}$$
$$MT = U^{\mathbf{M}}F^{\mathbf{M}}T \cong U^{\mathbf{M}}\bar{T}F^{\mathbf{M}}$$

it follows that $U^{\mathbf{M}}\widetilde{H}F^{\mathbf{M}} \cong U^{\mathbf{M}}\bar{T}F^{\mathbf{M}}$, that is, the images of $\widetilde{H}$ and $\bar{T}$ on free algebras share (up to bijection) the same underlying set. Taking into account that $HM \cong MT$ is an isomorphism of $\mathbf{M}$-algebras, we obtain $\widetilde{H} \cong \bar{T}$ on free algebras.

As M and T are finitary, $\bar{T}$ is determined by its action on finitely generated free algebras.

Since M is finitary, $U^{\mathbf{M}}$ creates filtered colimits; H being also finitary, it follows that $\widetilde{H}$ is finitary too. In particular, $\widetilde{H}$ is determined by its action

---

[6] The results in Proposition 3.3.i. hold without Assumption IV. But because in the remaining of this section we refer mostly to finitary functors, the referees suggested we emphasize this by an Assumption.

on all finitely presentable algebras. But every finitely presentable algebra is a reflexive coequalizer of finitely generated free algebras ([12]) and $\widetilde{H}$, being finitary, preserves such colimits. Therefore $\widetilde{H}$ will be determined by its action on finitely generated free algebras.

It follows that $\widetilde{H} \cong \bar{T}$ on all $\mathbf{M}$-algebras. $\square$

Example 3.4. Take $TX = 1 + A \times X$, with A finite set and $\mathbf{M}$ any Set-monad. Then a Kleisli lifting of T exists, namely for each map $X \xrightarrow{f} MY$, take $TX \xrightarrow{f} MTY$ to be the composite

$$\begin{aligned} &TX = 1 + A \times X \overset{1+A\times f}{\longrightarrow} 1 + A \times MY \longrightarrow \\ &1 + M(A \times Y) \longrightarrow M1 + M(A \times Y) \longrightarrow M(1 + A \times Y) \end{aligned}$$

where the map $1 + A \times MY \longrightarrow 1 + M(A \times Y)$ is obtained from the canonical strength of the monad, while $1 + M(A \times Y) \longrightarrow M1 + M(A \times Y)$ uses the unit of the monad and $M1 + M(A \times Y) \longrightarrow M(1 + A \times Y)$ comes from the coproduct property. Also, it is easy to see that the extension of T to $\mathbf{M}$-algebras is $\bar{T}X = F^{\mathbf{M}}1 + A \cdot X$, for each algebra X, where this time the coproduct (respectively the copower) is computed in $\mathsf{Alg}(\mathbf{M})$. If the category of $\mathbf{M}$-algebras has finite biproducts (as in the case of the monad induced by a semi-ring, see Example 2.14), then $\bar{T}$ is the lifting to $\mathsf{Alg}(\mathbf{M})$ of the Set-endofunctor $HX = M1 \times X^A$. Hence (T, H) form a commuting pair.

The motivation for studying commuting pairs appears clearly if we combine the previous proposition with our main result from Theorem 2.7, obtaining the following:

Corollary 3.5. *Assume the assumptions of Proposition 3.3.ii. hold. If H is ω^{op}-continuous and $M0 = 1$ as $\mathbf{M}$-algebras, then the final H-coalgebra is the completion of the free $\mathbf{M}$-algebra built on the initial T-algebra under a suitable metric.*

Proof. Follows from Theorem 2.7, by noticing that the M-image of the initial T-algebra (which exists as T is finitary, hence ω-cocontinuous) is the initial $\bar{T}$-algebra (by construction, $\bar{T}$ is finitary, so ω-cocontinuous), while H and $\widetilde{H}$ share same final coalgebra. $\square$

Example 3.6. We come back to the situation presented in Example 3.4 and take the monad induced by a semi-ring $\Bbbk$, as in Example 2.14. Then the initial T-algebra is A^*. The free $\mathbf{M}$-algebra on A^* is the direct sum of A^* copies of $\Bbbk$, that is, the polynomial $\Bbbk$-algebra in non-commuting A-variables $\Bbbk[A]$ (in the category of $\Bbbk$-semimodules), while the final H-coalgebra is $\Bbbk^{A^*}$, the non-commutative power series $\Bbbk$-algebra. The completion was described in Example 2.14.

The situation described until now in this section can be presented as follows: if two Set-endofunctors T and H are given, one may search for an appropriate monad such that (T, H) form a commuting pair. As there is a special bond between algebras of T and coalgebras of H, it is not clear whether the general case of any two (finitary) Set-endofunctors would have a solution. But there is another possible approach: start only with one endofunctor and additionally with a (finitary) monad; then find a distributive law inducing a Kleisli (or algebra) lift. Once this is accomplished, one should built a second endofunctor on Set (assuming this is possible) in order to obtain a commuting pair, using the functor obtained on $\mathsf{Alg}(\mathbf{M})$.

For lifting to the Kleisli category, there is the following suitable situation: for all commutative monads $\mathbf{M}$ and for all polynomial (more generally, for all analytic) functors T, a distributive law $TM \longrightarrow MT$ can always be constructed ([24]). The commutativity of $\mathbf{M}$ ensures also the existence of a tensor product $\otimes$ on $\mathsf{Alg}(\mathbf{M})$, such that the free functor $F^{\mathbf{M}} : (\mathsf{Set}, \times) \longrightarrow (\mathsf{Alg}(\mathbf{M}), \otimes)$ is strong monoidal ([14]). If the polynomial functor T is $TX = \coprod_{n \geq 0} \Sigma_n \times X^n$, an obvious choice of the Kleisli lift would give (the extension) $\bar{T}X = \oplus_{n \geq 0} F^{\mathbf{M}} \Sigma_n \otimes X^{\otimes n}$, where this time X denotes an $\mathbf{M}$-algebra and the coproduct $\oplus$ is computed in $\mathsf{Alg}(\mathbf{M})$. Assume now the monad is of effective descent type, or equivalently, that the free functor $F^{\mathbf{M}}$ is comonadic [7]. Then there exists a Set-endofunctor H such that (T, H) form a commuting pair (due to space limitations, the construction of H can be found in Appendix B).

Lifting functors to the category of algebras seems to be more problematic, even for the simplest case of polynomial functors (see paragraph B):

- Take $\mathbb{k}$ an algebra for a monad $\mathbf{M}$ and H the constant functor to $\mathbb{k}$. In this case, one may form a commuting pair if and only if $\mathbb{k}$ is a free $\mathbf{M}$-algebra. Then T is also a constant functor; in particular, Corollary 3.5 is trivially true.

- For $HX = \mathbb{k} \times X$, with $\mathbb{k}$ the carrier of an algebra for a monad $\mathbf{M}$, we make two additional assumptions: that the category $\mathsf{Alg}(\mathbf{M})$ has finite biproducts (for example if $\mathbb{k}$ is a semi-ring and $\mathbf{M}$ is the monad induced by it, as in Example 2.14) and that $\mathbb{k}$ is the carrier of a free algebra with set of generators B (if $\mathbb{k}$ is a semi-ring, then it is the free algebra built on the singleton). Then there is a commuting pair (T, H) where $TX = B + X$. The final H-coalgebra is the set of all streams on $\mathbb{k}$, while the $\mathbf{M}$-algebra on the initial T-algebra is the ω-copower of $MB \cong \mathbb{k}$.

- For $HX = X^A$, a power functor, the most convenient way of finding a commuting pair is again the existence of biproducts in $\mathsf{Alg}(\mathbf{M})$, this time A-indexed. For then the correspondent functor will be the Set-copower $TX = A \cdot X$. But in this case no relevant answers are obtained in

---

[7] This happens precisely when $2 \xrightarrow{u_2} M2$ is injective and $M0 \xrightarrow{M!} M1 \overset{M(inl)}{\underset{M(inr)}{\rightrightarrows}} M(2)$ is an equalizer. Examples of such monads are the lift monad, the multiset monad, the powerset monad and the subdistribution monad ([23]).

the initial-final (co)algebra relation, as both these objects are degenerate (empty initial T-algebra, singleton final H-coalgebra).

- If $H = MG$, for any Set-functor G, there is an obvious $\mathbf{M}$-commuting pair $(T = GM, H = MG)$. Assuming preservation of all (co)limits required by Corollary 3.5, we get the final MG-coalgebra exhibited as completion of the M-image of the initial GM-algebra.

4. Conclusions

Summary Given a functor $H : \mathsf{Set} \longrightarrow \mathsf{Set}$ and a monad $\mathbf{M}$ on Set, we have studied H-coalgebras where carriers have an $\mathbf{M}$-algebra structure. More precisely, we have considered the situation where H can be lifted to a functor $\widetilde{H}$ on $\mathsf{Alg}(\mathbf{M})$. In this case, the adjunction $F^{\mathbf{M}} \dashv U^{\mathbf{M}} : \mathsf{Alg}(\mathbf{M}) \longrightarrow \mathsf{Set}$ lifts to an adjunction $\widetilde{F} \dashv \widetilde{U} : \mathsf{Coalg}(\widetilde{H}) \longrightarrow \mathsf{Coalg}(H)$. In particular, we may say that the final $\widetilde{H}$-coalgebra is the final H-coalgebra equipped with an $\mathbf{M}$-algebra structure. Theorem 2.7 then states that the final $\widetilde{H}$-coalgebra is the completion of the initial $\widetilde{H}$-algebra. To further analyse the initial $\widetilde{H}$-algebra, we say that (T, H) form an $\mathbf{M}$-commuting pair of endofunctors if $MT \cong HM$. Corollary 3.5 states that in such a situation the initial $\widetilde{H}$-algebra coincides with the free $\mathbf{M}$-algebra of the initial T-algebra.

Future work If the functor H is not continuous (for example the finite powerset functor), then the final sequence has to be extended beyond ω steps. What does happen with the completion procedure on $\mathsf{Alg}(\mathbf{M})$ in such cases? We believe that an answer to this question is worth considering in the future.

The notion of a commuting pair of endofunctors with respect to a monad, defined in the second part of this paper, seems to be new; however, a detailed analysis and more examples are needed in order to better understand this structure (like the connection between bisimulations and traces). We plan to do this in a further paper.

Acknowledgments

The authors would like to thank the referees for their valuable suggestions, which helped improving the first version of this paper.

References

[1] J. Adámek, *Final coalgebras are ideal completions of initial algebras*, J. Logic Comput. **12**(2) (2002), 217-242.

[2] J. Adámek, *On final coalgebras of continuous functors*, Theor. Comput. Sci. **294**(1-2) (2003), 3-29.

[3] J. Adámek, H. Herrlich and G.-E. Strecker, "Abstract and Concrete Categories: the joy of cats", John Wiley & Sons, New York, 1990.

[4] J. Adámek, V. Koubek, *Are colimits of algebras simple to construct?*, J. Algebra **66**(1) (1980), 226-250.

[5] J. Adámek, J. Rosický, *On sifted colimits and generalized varieties*, Theory Appl. Categ. **8** (2001), 33-53.

[6] P. America, J. Rutten, *Solving reflexive domain equations in a category of complete metric spaces*, J. Comput. Syst. Sci. **39**(3) (1989), 343-375.

[7] A. Balan, A. Kurz, *On coalgebras over algebras*. In "Proceedings of the Tenth Workshop on Coalgebraic Methods in Computer Science (CMCS 2010)", Electr. Notes Theor. Comp. Sci. **264**(2) (2010), 47-62.

[8] M. Barr, *Algebraically compact functors*, J. Pure Appl. Algebra **82**(3) (1992), 211-231.

[9] M. Barr, *Terminal coalgebras in well-founded set theory*, Theor. Comput. Sci. **114**(2) (1993), 299-315.

[10] F. Borceux, "Handbook of Categorical Algebra", vol.1-3, Encycl. Mathem. Appl. **50-52**, Cambridge Univ. Press, Cambridge, 1994.

[11] A. Corradini, M. Große-Rhode and R. Heckel, *Structured transition system as lax coalgebras*. In "Proceedings of the first Workshop on Coalgebraic Methods in Computer Science CMCS'98", Electr. Notes Theor. Computer Sci. **11** (1998), 22-41.

[12] P. Gabriel, F. Ulmer, "Lokal präsentierbare Kategorien", Lect. Notes Math. **221**, Springer Verlag, Berlin, 1971.

[13] I. Hasuo, B. Jacobs and A. Sokolova, *Generic trace theory*. In "Proceedings of the Eighth Workshop on Coalgebraic Methods in Computer ScienceCMCS'06", Electr. Notes Theor. Computer Sci. **164**(1) (2006), 47-65.

[14] B. Jacobs, *Semantics of weakening and contraction*, Ann. Pure Appl. Logic **69**(1) (1994), 73-106.

[15] B. Jacobs, *Distributive laws for the coinductive solution of recursive equations*, Inform. and Comput. **204**(4) (2006), 561-587.

[16] B. Jacobs, *A bialgebraic review of deterministic automata, regular expressions and languages*. In "Algebra, Meaning and Computation", Lect. Notes Comput. Sci. **4060**, Springer Verlag, Berlin, 2006, 375-404.

[17] P. T. Johnstone, *Adjoint lifting theorems for categories of algebras*, Bull. London Math. Soc. **7**(3) (1975), 294-297.

[18] P. T. Johnstone, "Stone spaces", Cambr. Stud. Adv. Math. **3**, Cambridge Univ. Press, Cambridge, 1982.

[19] C. Kissig and A. Kurz, *Generic trace logics*, arXiv:1103.3239v1.

[20] F.E.J. Linton, *Coequalizers in categories of algebras.* In "Seminar on Triples and Categorical Homology Theory", Lect. Notes Math. **80**, Springer Verlag, 1969, 75-90.

[21] S. Mac Lane, "Categories for the working mathematician", 2nd ed., Grad. Texts Math. **5**, Springer Verlag, New York, 1998.

[22] E. Manes, *Monads of sets.* In "Handbook of algebra", vol. 3, North-Holland, Amsterdam, 2003, 67-153.

[23] B. Mesablishvili, *Monads of effective descent type and comonadicity*, Th. Appl. Categ. **16**(1) (2006), 1-45.

[24] St. Milius, Th. Palm and D. Schwencke, *Complete iterativity for algebras with effects.* In "Algebra and Coalgebra in Computer Science", Lect. Notes Comput. Sci. **5728**, Springer Verlag, Berlin, 2009, 34-48.

[25] L. Moss, *A note on expressive coalgebraic logics for finitary set functors*, J. Logic Comput. **20**(5) 2010, 1101-1111.

[26] J. Rutten, *Automata and coinduction (an exercise in coalgebra).* In "CONCUR'98: concurrency theory", Lect. Notes Comput. Sci. **1466**, Springer Verlag, 1998, 194-218.

[27] J. Rutten, *Behavioural differential equations: A coinductive calculus of streams, automata, and power series*, Theor. Comput. Sci. **308**(1-3), 2003, 1-53.

[28] J. Rutten, *Coinductive counting with weighted automata*, J. Autom. Lang. Comb. **8**(2) (2003), 319-352.

[29] H. H. Schaefer, "Topological vector spaces", Grad. Texts Math. **3**, Springer Verlag, New York, 1971.

[30] M. B. Smyth, G. D. Plotkin, *The category-theoretic solution of recursive domain equations*, SIAM J. Comput. **11** (1982), 761-783.

[31] D. Turi, G. D. Plotkin, *Towards a mathematical operational semantics.* In "Proceedings of the 12th Annual IEEE Symposium on Logic in Computer Science", LICS, IEEE Computer Society, Washington, DC, 1997, 280-291.

[32] D. H. Van Osdol, *Sheaves in regular categories.* In "Exact categories and categories of sheaves", Lect. Notes Math. **236**, Springer Berlin/Heidelberg, 1971, 223-239.

[33] J. Worrell, *On the final sequence of a finitary set functor*, Theor. Comput. Sci. **338**(1-3) (2005), 184-199.

Appendix A.

Proposition. *Let $\mathcal{C}$ be any category, $\mathbf{M} = (M, m, u)$ a monad on $\mathcal{C}$ and $H : \mathcal{C} \longrightarrow \mathcal{C}$ a functor that admits a lifting $\hat{H}$ to $Kl(\mathbf{M})$. Assume $\mathsf{Alg}(\mathbf{M})$ has coequalizers and denote by $\mathcal{I} : Kl(\mathbf{M}) \longrightarrow \mathsf{Alg}(\mathbf{M})$ the comparison functor. Then the left Kan extension $\bar{H}$ of $\mathcal{I}\hat{H}$ along $\mathcal{I}$ exists and the universal arrow associated to it is a natural isomorphism.*

Proof. For $\mathcal{C}$ a cocomplete category, $\mathsf{Alg}(\mathbf{M})$ having coequalizers implies that it is cocomplete ([20]); if $\mathcal{C}$ is also small, the left Kan extension exists with the universal arrow being an isomorphism, as $\mathcal{I}$ is full and faithful. However, for general $\mathcal{C}$ this argument does not apply; we shall construct the functor $\bar{H}$ and the natural transformation $\zeta : \mathcal{I}\hat{H} \longrightarrow \bar{H}\mathcal{I}$ associated to the Kan extension "by hand".

We shall denote by $\varsigma : HM \longrightarrow MH$ the distributive law corresponding to the lifting (8). We recall the construction of $\hat{H}$: on objects $X \in \mathrm{Ob}(Kl(\mathbf{M})) = \mathrm{Ob}(\mathcal{C})$, $\hat{H}X = HX$, and on arrows, $\hat{H}(X \xrightarrow{f} MY) = HX \xrightarrow{Hf} HMY \xrightarrow{\varsigma_Y} MHY$. The comparison functor $\mathcal{I} : Kl(\mathbf{M}) \longrightarrow \mathsf{Alg}(\mathbf{M})$ is given by $\mathcal{I}X = (MX, m_X)$ and $\mathcal{I}(X \xrightarrow{f} MY) = MX \xrightarrow{Mf} M^2Y \xrightarrow{m_Y} MY$. It is straightforward to see that their composition will be: $\mathcal{I}\hat{H}X = (MHX, m_{HX})$ with

$$\mathcal{I}\hat{H}(X \xrightarrow{f} MY) = \begin{tikzcd} MHX \arrow[r, "MHf"] & MHMY \arrow[r, "M\varsigma_Y"] & M^2HY \arrow[r, "m_{HY}"] & MHY \end{tikzcd}$$

In order to built the functor part of the left Kan extension, we start with an $\mathbf{M}$-algebra $(A, MA \xrightarrow{a} A)$ and write it as a coequalizer of free algebras in $\mathsf{Alg}(\mathbf{M})$:

$$\begin{tikzcd} M^2A \arrow[r, shift left, "Ma"] \arrow[r, shift right, "m_A"'] & MA \arrow[r, "a"] & A \end{tikzcd} \tag{A.1}$$

Define now $\bar{H}A$ as the coequalizer in $\mathsf{Alg}(\mathbf{M})$ (which exists by hypothesis) of the pair $(MHa, m_A \circ M\varsigma_A)$, namely $\begin{tikzcd} MHMA \arrow[r, shift left, "MHa"] \arrow[r, shift right, "m_A \circ M\varsigma_A"'] & MHA \arrow[r, "\pi_A"] & \bar{H}A \end{tikzcd}$. It is immediate that this defines a functor on $\mathsf{Alg}(\mathbf{M})$. In particular, this construction can be applied to free algebras. But in this situation the corresponding parallel pair of arrows leads to a split coequalizer in $\mathsf{Alg}(\mathbf{M})$, given by

$$\begin{tikzcd} MHM^2X \arrow[r, shift left, "MHm_X"] \arrow[r, shift right, "m_{HMX} \circ M\varsigma_{MX}"'] & MHMX \arrow[r, "m_{HX} \circ M\varsigma_X"] \arrow[l, bend left=50, "MHMu_X"] & MHX \arrow[l, bend left=50, "MHu_X"] \end{tikzcd} \tag{A.2}$$

It follows that $\bar{H}\mathcal{I}X = MHX$ on objects. To see the behavior of $\bar{H}\mathcal{I}$ on arrows, start with $X \xrightarrow{f} MY$ in $\mathcal{C}$; in the diagram below,

$$\begin{tikzcd}
MHM^2X \arrow[r, shift left, "MHm_X"] \arrow[r, shift right, "m_{HMX}\circ M\varsigma_{MX}"'] \arrow[d, "MHM^2f"'] & MHMX \arrow[r, "m_{HX}\circ M\varsigma_X"] \arrow[d, "MHMf"] & MHX \arrow[d, "MHf"] \\
MHM^3Y \arrow[r, shift left, "MHm_{MY}"] \arrow[r, shift right, "m_{HM^2Y}\circ M\varsigma_{M^2Y}"'] \arrow[d, "MHMm_Y"'] & MHM^2Y \arrow[r, "m_{HMY}\circ M\varsigma_{MY}"] \arrow[d, "MHm_Y"] & MHMY \arrow[d, "m_{HY}\circ M\varsigma_Y"] \\
MHM^2Y \arrow[r, shift left, "MHm_Y"] \arrow[r, shift right, "m_{HMY}\circ M\varsigma_{MY}"'] & MHMY \arrow[r, "m_{HY}\circ M\varsigma_Y"] & MHY
\end{tikzcd}$$

all three rows are split coequalizers as in (A.2); the left upper and lower squares commute serially by naturality of m and ς and associativity of m, the upper right square commutes again by naturality, while for the lower right square, one needs to use also the properties of ς as a distributive law.

Hence $\bar{H}\mathcal{I}$ coincides with $\mathcal{I}\hat{H}$. To check that indeed $\bar{H}$ is the left Kan extension of $\mathcal{I}\hat{H}$ along $\mathcal{I}$, with identity as the associated natural transformation $\mathcal{I}\hat{H} \longrightarrow \bar{H}\mathcal{I}$, take $G : \mathsf{Alg}(\mathbf{M}) \to \mathsf{Alg}(\mathbf{M})$ a functor, together with natural transformation $\zeta : \mathcal{I}\hat{H} \longrightarrow G\mathcal{I}$. This means that we have algebra arrows $\zeta_X : MHX \longrightarrow GMX$ for X any object of $\mathcal{C}$ such that for any $X \xrightarrow{f} MY$, the diagram below commutes:

$$\begin{tikzcd}
MHX \arrow[r, "MHf"] \arrow[d, "\zeta_X"'] & MHMY \arrow[r, "M\varsigma_Y"] & M^2HY \arrow[r, "m_{HY}"] & MHY \arrow[d, "\zeta_Y"] \\
GMX \arrow[r, "GMf"] & GM^2Y \arrow[rr, "Gm_Y"] & & GMY
\end{tikzcd} \tag{A.3}$$

For a given algebra (A, a), apply the above (A.3) to $MA \xrightarrow{Id} MA$ to obtain

$$Gm_A \circ \zeta_{MA} = \zeta_A \circ m_{HA} \circ M\varsigma_A \tag{A.4}$$

and to $MA \xrightarrow{a} A \xrightarrow{u_A} MA$ for

$$GMa \circ \zeta_{MA} = \zeta_A \circ MHa \tag{A.5}$$

Now consider the following diagram:

$$\begin{tikzcd}
MHMA \arrow[r, shift left, "MHa"] \arrow[r, shift right, "m_A\circ M\varsigma_A"'] \arrow[d, "\zeta_{MA}"'] & MHA \arrow[r, "\pi_A"] \arrow[d, "\zeta_A"] & \bar{H}A \arrow[d, dashed, "\theta_A"] \\
GM^2A \arrow[r, shift left, "GMa"] \arrow[r, shift right, "Gm_A"'] & GMA \arrow[r, "Ga"] & GA
\end{tikzcd} \tag{A.6}$$

where the second row is obtained by applying G to (A.1). We have

$$\begin{aligned}
Ga \circ \zeta_A \circ MHa &= Ga \circ GMa \circ \zeta_{MA} \text{ by (A.5)} \\
&= Ga \circ Gm_A \circ \zeta_{MA} \text{ by (A.1)} \\
&= Ga \circ \zeta_A \circ m_{HA} \circ M\varsigma_A \text{ by (A.4)}
\end{aligned}$$

hence by the coequalizer property of the first row in (A.6) there is a unique algebra map $\theta_A : \bar{H}A \longrightarrow GA$. It is easy to see that θ is natural. A similar argument as above shows that on free algebras, θ_{MX} is precisely ζ_X, which is the same as the commutativity of

$$\begin{tikzcd}
\mathcal{I}\hat{H} \arrow[r, "Id"] \arrow[dr, "\zeta"'] & \bar{H}\mathcal{I} \arrow[d, "\theta_{\mathcal{I}}"] \\
& G\mathcal{I}
\end{tikzcd}$$

By construction, θ is unique with this property. We have thus obtained the desired left Kan extension. □

Appendix B.

Let $\mathbf{M}$ be a commutative finitary Set-monad of effective descent type and T a polynomial Set-functor, $TX = \coprod_{n\geq 0} \Sigma_n \times X^n$. Let $\bar{T}X = \oplus_{n\geq 0} M\Sigma_n \otimes X^{\otimes n}$ be an extension of T to $\mathsf{Alg}(\mathbf{M})$.

In this appendix we shall see how to build a Set-functor H such that $HU^{\mathbf{M}} \cong U^{\mathbf{M}}\bar{T}$. Consequently, (T, H) will form a commuting pair.

Recall first how both the the tensor product (which exist as the monad is commutative) and the coproduct on $\mathsf{Alg}(\mathbf{M})$ are built as reflexive coequalizers of free $\mathbf{M}$-algebras (see for example [14], Lemma 3.2 and Lemma 5.1): for any two $\mathbf{M}$-algebras $MX \xrightarrow{x} X$ and $MY \xrightarrow{y} Y$, we have

$$\begin{tikzcd}
M(MX \times MY) \arrow[r, shift left, "m_{X\times Y}\circ M(\varphi_{2,X,Y})"] \arrow[r, shift right, "M(x\times y)"'] & M(X \times Y) \arrow[l, bend left=40, "M(u_X\times u_Y)"] \arrow[r] & X \otimes Y
\end{tikzcd}$$

where $\varphi_{2,X,Y} : MX \times MY \longrightarrow M(X \times Y)$ is the monoidal structure map of the monad. Next, for any (countable) family of algebras $(MX_n \xrightarrow{x_n} X_n)_n$, their coproduct is the following reflexive coequalizer in $\mathsf{Alg}(\mathbf{M})$:

$$\begin{tikzcd}
M(\coprod\limits_n MX_n) \arrow[r, shift left, "m_{\coprod\limits_n X_n}\circ M(\tau)"] \arrow[r, shift right, "M(\coprod\limits_n x_n)"'] & M(\coprod\limits_n X_n) \arrow[l, bend left=40, "M(\coprod\limits_n u_{X_n})"] \arrow[r] & \oplus_n X_n
\end{tikzcd}$$

Here $\coprod$ denotes the coproduct in Set and $\tau : \coprod\limits_n MX_n \longrightarrow M(\coprod\limits_n X_n)$ is the canonical arrow from the coproduct.

In the sequel, we shall implicitly use that $F^{\mathbf{M}}$ preserves any colimits and $U^{\mathbf{M}}$ preserves reflexive coequalizers ($\mathbf{M}$ being finitary).

Then we can write down the diagram (B.1):

$$\begin{tikzcd}
M(\coprod_n M^2(M^2\Sigma_n \times (MX)^n)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M(\coprod_n M(M^2\Sigma_n \times (MX)^n)) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & M(\coprod_n (M^2\Sigma_n \times (MX)^n)) \arrow[d, dotted, shift left] \arrow[d, dotted, shift right] \\
M(\coprod_n M^2(M\Sigma_n \times X^n)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & M(\coprod_n M(M\Sigma_n \times X^n)) \arrow[r] \arrow[d] & M(\coprod_n (M\Sigma_n \times X^n)) \arrow[d, dotted] \\
M(\coprod_n M(M\Sigma_n \otimes X^{\otimes n})) \arrow[r, shift left] \arrow[r, shift right] & M(\coprod_n (M\Sigma_n \otimes X^{\otimes n})) \arrow[r] & \bigoplus_n (M\Sigma_n \otimes X^{\otimes n})
\end{tikzcd} \tag{B.1}$$

$$\begin{tikzcd}
M^2(\coprod_n (M^2\Sigma_n \times (MX)^n)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "m\circ\coprod_n (Id\times(Mu)^n)"] & M^2(\coprod_n (M\Sigma_n \times X^n)) \arrow[r] \arrow[d, "m\circ\coprod_n (Id\times u^n)"] & G\bar{T}X = M(\bigoplus_n (M\Sigma_n \otimes X^{\otimes n})) \arrow[d, dotted] \\
M(\coprod_n (M^2\Sigma_n \times (M^2X)^n)) \arrow[r, shift left] \arrow[r, shift right] & M(\coprod_n (M\Sigma_n \times (MX)^n)) \arrow[r] & \bar{T}GX = \bigoplus_n (M\Sigma_n \otimes (MX)^{\otimes n})
\end{tikzcd} \tag{B.2}$$

where $\bar{T}X$ is in the lower right-hand corner and the last horizontal row is the reflexive coequalizer in $\mathsf{Alg}(\mathbf{M})$ giving the coproduct $\oplus$; the first two vertical rows are obtained by applying $M\coprod_n M$ (actually, is $F^{\mathbf{M}}\coprod_n U^{\mathbf{M}}F^{\mathbf{M}}U^{\mathbf{M}}$), respectively $M\coprod_n$, to the reflexive coequalizers in $\mathsf{Alg}(\mathbf{M})$ giving the tensor products $M\Sigma_n \otimes X^{\otimes n}$ for all $n \geq 0$. As $\mathbf{M}$ is finitary, these vertical rows will still be coequalizers in $\mathsf{Alg}(\mathbf{M})$. Writing $M^2\Sigma_n \times (MX)^n$ and $M\Sigma_n \times X^n$ as coequalizers of free algebras and applying $M\coprod_n$, we get the first two horizontal rows.

Summing up, all three horizontal rows and first two vertical rows are reflexive coequalizers in $\mathsf{Alg}(\mathbf{M})$ and all squares commute conveniently. Consequently, we can fill the last vertical row with arrows such that it becomes also a reflexive coequalizer, namely

$$M\Big(\coprod_n (M^2\Sigma_n \times (MX)^n)\Big) \rightrightarrows M\Big(\coprod_n (M\Sigma_n \times X^n)\Big) \longrightarrow \bigoplus_n (M\Sigma_n \otimes X^{\otimes n}) \tag{B.3}$$

Denote by $\mathbf{G}$ the comonad associated with the adjunction $F^{\mathbf{M}} \dashv U^{\mathbf{M}}$. We have the following picture, with $C_{\mathbf{G}}$ the cofree functor, $V_{\mathbf{G}}$ the forgetful one and K the comparison functor:

$$\begin{tikzcd}
\mathsf{Alg}(\mathbf{M}) \arrow[r, bend right, "C_{\mathbf{G}}"'] \arrow[r, phantom, "\perp"] \arrow[d, bend left, "U^{\mathbf{M}}"] & \mathsf{Coalg}(\mathbf{G}) \arrow[l, bend right, "V_{\mathbf{G}}"'] \\
\mathsf{Set} \arrow[u, bend left, "F^{\mathbf{M}}"] \arrow[u, phantom, "\dashv"] \arrow[ur, bend right, "K"'] &
\end{tikzcd}$$

Dually to the situation described at the beginning of Section 3, liftings of $\bar{T}$ to the Kleisli category of this comonad are in one-to-one correspondence with natural transformations $G\bar{T} \longrightarrow \bar{T}G$ satisfying some commutative diagrams (dual to (8)). In order to obtain such a natural transformation, we use (B.3) twice, with connecting homomorphisms as in diagram (B.2). After some tedious computations, it follows that there is a unique arrow $G\bar{T} \longrightarrow \bar{T}G$ which makes the right square of (B.2) commute and allows for a lifting of $\bar{T}$ as explained before.

As by hypothesis the monad $\mathbf{M}$ is of effective descent type, the category of coalgebras for the associated comonad $\mathbf{G}$ is equivalent to Set. Consequently, $\mathsf{Coalg}(\mathbf{G})$ has all equalizers. We can dualize now the Proposition of Appendix A to obtain an extension of $\bar{T}$ to $\mathbf{G}$-coalgebras (a functor T_1 such that $T_1 C \cong C\bar{T}$).

Recall that by hypothesis, K is an equivalence of categories; denote by K^{-1} its inverse and define a Set-endofunctor by $H = K^{-1}T_1K$. Then $HU^{\mathbf{M}} \cong U^{\mathbf{M}}\bar{T}$.