Summary

1. Coquasi-Hopf algebras and their coactions
2. Galois extensions and structure theorems for relative Hopf modules
   1. Cleft extensions
3. Galois extensions and a Hopf algebroid construction
(Majid (1992), Panaite, Ştefan (1997)) $H$ coquasi-Hopf algebra

- coassociative coalgebra $\Delta$, $\epsilon$
- $\exists$ unit and multiplication, no longer associative
- associativity of multiplication controlled by $\omega \in (H \otimes H \otimes H)^*$
- $\exists$ antimorphism of coalgebras (antipode) $S$, elements $\alpha, \beta \in H^*$
Coquasi-Hopf algebras and their coactions

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- **Monoidal category** (but not strict!) of comodules \((\mathcal{M}^H, \otimes)\)
$H$ coquasi-Hopf algebra.

- A right $H$-comodule algebra $A$ is an algebra in the monoidal category $\mathcal{M}^H$.
- $A$ is no longer associative!
- The space of coinvariants $B = A^{coH} = \{ a \in A | \rho_A(a) = a \otimes 1_H \}$ is an algebra (associative!)
Coquasi-Hopf algebras and their coactions

- (Bulacu, Nauwelaerts, 2000) Right relative \((H, A)\)-Hopf module is a right \(A\)-module in \(\mathcal{M}^H\).
- The category of relatives Hopf modules \(\mathcal{M}^H_A\)
- Adjunction of categories: \(\mathcal{M}_B \leftrightarrow \mathcal{M}^H_A\) with counit \(\varepsilon_M\)

\[
(-) \otimes_B A \\
(-) ^{coH}
\]
**Twist invariance**: take $\tau$ twist on $H$.
Then $H_\tau$ is a coquasi-Hopf algebra, with new multiplication $g \cdot_\tau h = \tau(g_1, h_1) g_2 h_2 \tau^{-1}(g_3, h_3)$, same unit, new cocycle $A_{\tau^{-1}}$ is a comodule algebra over $H_\tau$, but with new operation $g \circ_\tau h = g_1 h_1 \tau^{-1}(g_2, h_2)$

Category isomorphism $\mathcal{M}_A^H \simeq \mathcal{M}_{A_{\tau^{-1}}}^{H_\tau}$
Easy way to produce comodule algebras:
Start with $H$ Hopf algebra. Then $A = H$ is a comodule algebra over itself. Take now $\tau$ twist on $H$. Then $A_{\tau^{-1}}$ is comodule algebra over coquasi-Hopf algebra $H_{\tau}$.
Coquasi-Hopf algebras and their coactions

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- Apply to the Hopf algebra $H = \mathbb{k}G$, for $G$ finite group, and twist induced by a 2-cocycle $\sigma : G \times G \to \mathbb{k}^*$
Easy way to produce comodule algebras:
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Apply to the Hopf algebra $H = kG$, for $G$ finite group, and twist induced by a 2-cocycle $\sigma : G \times G \longrightarrow k^*$

For $G = (\mathbb{Z}_2)^n$: all Cayley and Clifford algebras as comodule algebras over some coquasi-Hopf algebra (Albuquerque, Majid 1998-2000)
Galois extensions and structure theorems for relative Hopf modules

$H$ coquasi-Hopf algebra, $A$ right comodule algebra, $B = A^{coH}$

**Definition (Balan, 2007)**

Galois extension $B \subseteq A \iff$ bijective map

\[
\text{can} : \quad A \otimes_B A \longrightarrow A \otimes H
\]

\[
a \otimes_B b \quad \longrightarrow \quad a_0 b_0 \otimes \omega^{-1}(a_1, b_1 \beta(b_2), S(b_3)) b_4
\]
Galois extensions and structure theorems for relative Hopf modules

Example (Masuoka, 2003)

From operator algebras theory
Matched pair of finite groups $F, G \rightarrow\rightarrow$ construction of a coquasi-Hopf algebra $H = \hat{G} \#_{\sigma, \tau} CF$ as a bicrossproduct with some cocycle data $(\omega, \sigma, \tau)$
Outer action of the matched pair on the hyperfinite $II_1$ factor $\mathcal{R} \rightarrow\rightarrow$ Galois extension $\mathcal{R}(\alpha, \nu_0) \subseteq \mathcal{R}(\beta, G)$ using classical definition
$a \otimes_B b \rightarrow ab_0 \otimes b_1$

In this case our formula $a \otimes_B b \rightarrow a_0 b_0 \otimes \omega^{-1}(a_1, b_1 \beta(b_2), S(b_3))b_4$
reduces to $a \otimes_B b \rightarrow ab_0 \otimes b_1$
Galois extensions and structure theorems for relative Hopf modules

- $G$ group, $\omega$ invertible normalized 3-cocycle
  Coquasi-Hopf algebra $H = (\mathbb{k} G, \omega)$
  A comodule algebra $\iff$ quasialgebra $G$-graded (Albuquerque, Majid, 1998-2000)
  $A_e \subseteq A$ is Galois $\iff$ $A$ is strongly graded
Galois extensions and structure theorems for relative Hopf modules

- $G$ group, $\omega$ invertible normalized 3-cocycle
  Coquasi-Hopf algebra $H = (\mathbb{k}G, \omega)$
  A comodule algebra $\iff$ quasialgebra $G$-graded (Albuquerque, Majid, 1998-2000)
  $A_e \subseteq A$ is Galois $\iff$ $A$ is strongly graded

- **Twist invariance:** for $\tau$ twist on $H$, it follows that $B \subseteq A$ is $H$-Galois $\iff$ $B \subseteq A_{\tau^{-1}}$ is $H_{\tau}$-Galois
Galois extensions and structure theorems for relative Hopf modules

For Hopf algebras: $can = \varepsilon_{A \otimes H}$, where $A \otimes H^* \in \mathcal{M}^H_A$

**Lemma**

$S$ bijective $\iff$ isomorphism of right $H$-comodules $H^* \otimes A^* \rightarrow A \otimes H^*$

**Proposition**

$A \otimes H$ becomes a right $A$-module in $\mathcal{M}^H$

Counit $\varepsilon_{A \otimes H} = \text{can}_f$, where $f = \text{twist induced by } S^{-1}$

$\varepsilon$ bijective $\iff$ Galois extension $B \subseteq A$
Cleft extensions

**Definition**

A right $H$-comodule algebra is **cleft** if $\exists \gamma, \delta : H \longrightarrow A$ such that

\[
\rho(\gamma(h)) = \gamma(h_1) \otimes h_1 \quad \rho(\delta(h)) = \delta(h_2) \otimes S(h_1)
\]

\[
\delta(h_1)\gamma(h_2) = \alpha(h)1_A \quad \gamma(h_1)\beta(h_2)\delta(h_3) = \varepsilon(h)1_A
\]
Cleft extensions

Proposition

There is a Morita context $\mathcal{M}(A)$ with

- rings $\text{Hom}^H(\overline{H}, A)$ and $\text{Hom}(H, B)$
- bimodules $\text{Hom}^H(H, A)$ and $\text{Hom}^H(H^S, A)$
- connecting morphisms

\[
\begin{align*}
(\cdot, \cdot) & : \text{Hom}^H(H, A) \otimes_{\text{Hom}^H(\overline{H}, A)} \text{Hom}^H(H^S, A) \longrightarrow \text{Hom}(H, B) \\
(p, q)(h) &= p(h_1) \beta(h_2) q(h_3) \\
[\cdot, \cdot] & : \text{Hom}^H(H^S, A) \otimes_{\text{Hom}(H, B)} \text{Hom}^H(H, A) \longrightarrow \text{Hom}^H(\overline{H}, A) \\
[q, p](h) &= q(h_1) p(h_2)
\end{align*}
\]
Cleft extensions

Theorem

$H$ coquasi-Hopf algebra, $A$ right $H$-comodule algebra $\implies$

1. First Morita map $[,]$ is surjective $\iff$
   \[
   \begin{cases}
   B \subseteq A \text{ is Galois} \\
   \exists \ n > 0, \text{ s. t. } A \text{ is direct summand in } (\cdot B \otimes H^\bullet)^n
   \end{cases}
   \]

2. Strict Morita context $\iff$
   \[
   \begin{cases}
   B \subseteq A \text{ is Galois} \\
   \exists \ n > 0, \text{ s. t. } A \text{ is direct summand in } (\cdot B \otimes H^\bullet)^n \\
   \exists \ n, \ r > 0 \text{ s. t. } A \text{ is direct summand in } (\cdot B \otimes H^\bullet)^n \text{ and } \\
   \cdot B \otimes H^\bullet \text{ is direct summand in } A^r
   \end{cases}
   \]

3. $B \subseteq A$ cleft extension $\implies$ strict Morita context.
Cleft extensions

Theorem

$H$ coquasi-Hopf algebra with bijective antipode, $A$ comodule algebra, $B = A^{coH}$. TFAE:

1. $B \subseteq A$ cleft
2. $\varepsilon_M$ bijective for all $M \in \mathcal{M}_A^H$ and $B \subseteq A$ has the normal basis property
3. $B \subseteq A$ Galois and $B \subseteq A$ has the normal basis property.

Then $\mathcal{M}_B \simeq \mathcal{M}_A^H$
Theorem

$H$ coquasi-Hopf algebra with bijective antipode, $A$ comodule algebra and $B = A^{coH}$. TFAE:

1. Existe a total integral $\gamma : H \to A$ (comodule map with $\gamma(1_H) = 1_A$) and $\text{can} : A \otimes_B A \to A \otimes H$ is surjective

2. The coinvariants functor $(-)^{coH}$ and the induction functor $- \otimes_B A$ form an equivalence of categories $M^H_A \simeq M_B$

3. (Left version) The coinvariants functor $(-)^{coH}$ and the induction functor $A \otimes_B -$ form an equivalence of categories $A M^H \simeq_B M$

4. $A$ is left $B$-module faithfully flat and $B \subseteq A$ is Galois

5. $A$ is right $B$-module faithfully flat and $B \subseteq A$ is Galois
Galois extensions and a Hopf algebroid construction

- $H$ coquasi-Hopf algebra, $A$ right $H$-comodule algebra
- $A^\bullet \otimes A^\bullet$ is a right comodule by the codiagonal coaction
  $\rho(a \otimes b) = a_0 \otimes b_0 \otimes a_1 b_1$
- The space of coinvariants $L = (A \otimes A)^{coH}$ is an associative $B^{op}$-algebra with unit $1_A \otimes 1_A$ and multiplication

$$
(a \otimes b)(c \otimes d) = a_0 c_0 \otimes d_0 b_0 \omega^{-1}(a_1, c_1, d_1 b_1) \omega(c_2, d_2, b_2)
$$
Galois extensions and a Hopf algebroid construction

- $H$ coquasi-Hopf algebra with bijective antipode, $A$ comodule algebra which is Galois left faithfully flat $B$-module

Category equivalence $\mathcal{M}_A^H \simeq \mathcal{M}_B$

$\mathcal{M}_A^H$ has a natural left $\mathcal{M}_A^H$-action: $\diamond : \mathcal{M}_A^H \times \mathcal{M}_A^H \rightarrow \mathcal{M}_A^H$, $V \diamond M = V \otimes M$, with structures $\rho(v \otimes m) = v_0 \otimes m_0 \otimes v_1 m_1$ and $(v \otimes m)a = v \otimes ma$

- It follows that $\mathcal{M}_B$ is a left $\mathcal{M}_A^H$-category, with structure:

$$\diamond : \mathcal{M}_A^H \times \mathcal{M}_B \rightarrow \mathcal{M}_B \quad V \diamond N = [V \otimes (N \otimes_B A_\bullet)]^{coH}$$

and coassociator $V \diamond (W \diamond N) \xrightarrow{\Psi_{V,W,M}^{-1}} (V \otimes W) \diamond N$, $\Psi_{V,W,M}^{-1}(v \otimes \{[w \otimes (n \otimes_B a)] \otimes_B b\}) = (v \otimes w) \otimes (n \otimes_B ab)$
A algebra in $\mathcal{M}^{\mathcal{H}} \hookrightarrow$ left $A$-modules within $\mathcal{M}^{\mathcal{H}}_A$ and $\mathcal{M}_B$

Obtain equivalent categories $A\mathcal{M}^{\mathcal{H}}_A$ and $A(\mathcal{M}_B)$

Recover our algebra $L = A\Diamond B^{op}$ and have category isomorphism $A(\mathcal{M}_B) \simeq A\Diamond B^{op}\mathcal{M}$ (Hopf algebra case: Schauenburg, 2003)

**Proposition**

$L\mathcal{M}$ is a monoidal category and $L$ becomes a $B^{op}$-Hopf algebroid.
Further step in constructing Hopf algebroids

- $H$ coquasi-Hopf algebra, $A$ Galois faithfully flat $\mathbb{k}$-module
Further step in constructing Hopf algebroids

- $H$ coquasi-Hopf algebra, $A$ Galois faithfully flat $k$-module
- Then $A$ is a commutative algebra in the category of Yetter-Drinfeld modules
Further step in constructing Hopf algebroids

- $H$ coquasi-Hopf algebra, $A$ Galois faithfully flat $k$-module
- Then $A$ is a commutative algebra in the category of Yetter-Drinfeld modules
- Some smash product of $A$ with $H$ should be a Hopf algebroid (for Hopf algebras - Militaru, Brzèzinski, 2001)