Positive Fragments of Coalgebraic Logics

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Abstract. Positive modal logic was introduced in an influential 1995 paper of Dunn as the positive fragment of standard modal logic. His completeness result consists of an axiomatization that derives all modal formulas that are valid on all Kripke frames and are built only from atomic propositions, conjunction, disjunction, box and diamond.
In this paper, we provide a coalgebraic analysis of this theorem, which not only gives a conceptual proof based on duality theory, but also generalizes Dunn's result from Kripke frames to coalgebras of weak-pullback preserving functors.
For possible application to fixed-point logics, it is note-worthy that the positive coalgebraic logic of a functor is given not by all predicate-liftings but by all monotone predicate liftings.

Keywords: coalgebraic logic, duality, positive modal logic

1 Introduction

Consider modal logic as given by atomic propositions, Boolean operations, and a unary box, together with its usual axiomatisation stating that box preserves finite meets. In [11], Dunn answered the question of an axiomatisation of the positive fragment of this logic, where the positive fragment is given by atomic propositions, lattice operations, and unary box and diamond.

Here we seek to generalize this result from Kripke frames to coalgebras for a weak pullback preserving functor. Whereas Dunn had no need to justify that the positive fragment actually *adds* a modal operator (the diamond), the general situation requires a conceptual clarification of this step. And, as it turns out, what looks innocent enough in the familiar case is at the heart of the general construction.

In the general case, we start with a functor $T : \mathsf{Set} \to \mathsf{Set}$. From T we can obtain by duality a functor $L : \mathsf{BA} \to \mathsf{BA}$ on the category BA of Boolean

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algebras, so that the free L-algebras are exactly the Lindenbaum algebras of the modal logic. We are going to take the functor L itself as the category theoretic counterpart of the corresponding modal logic. How should we construct the positive T-logic? Dunn gives us a hint in that he notes that in the same way as standard modal logic is given by algebras over BA, positive modal logic is given by algebras over the category DL of (bounded) distributive lattices. It follows that the positive fragment of (the logic corresponding to) L should be a functor $L' : \mathsf{DL} \to \mathsf{DL}$ which, in turn, by duality, should arise from a functor $T' : \mathsf{Pos} \to \mathsf{Pos}$ on the category Pos of posets and monotone maps.

The centre-piece of our construction is now the observation that any finitary-functor $T : \mathsf{Set} \to \mathsf{Set}$ has a canonical extension to a functor $T' : \mathsf{Pos} \to \mathsf{Pos}$. Theorem 4.11 then shows that this construction $T \mapsto T' \mapsto L'$ indeed gives the positive fragment of L and so generalizes Dunn's theorem.

An important observation about the positive fragment is the following: given any Boolean formula, we can rewrite it as a positive formula with negation only appearing on atomic propositions. In other words, the translation β from positive logic to Boolean logic given by

$$\beta(\Diamond\phi) = \neg\Box\neg\beta(\phi) \tag{1}$$

$$\beta(\Box\phi) = \Box\beta(\phi) \tag{2}$$

induces a bijection (on equivalence classes of formulas taken up to logical equivalence). More algebraically, we can formulate this as follows.

Given a Boolean algebra $B \in \mathsf{BA}$, let LB be the free Boolean algebra generated by $\{\Box b \mid b \in B\}$ modulo the axioms of modal logic. Given a distributive lattice A, let $L'A$ be the free distributive lattice generated by $\{\Box\phi : \phi \in A\} \cup \{\Diamond\phi \mid \phi \in A\}$ modulo the axioms of positive modal logic. Further, let us denote by $W : \mathsf{BA} \to \mathsf{DL}$ the forgetful functor. Then the above observation that every modal formula can be written, up to logical equivalence, as a positive modal formula with negations pushed to atoms, can be condensed into the statement that the (natural) distributive lattice homomorphism

$$\beta_B : L'WB \to WLB \tag{3}$$

induced by (1), (2) is an isomorphism.

Our main results are the following. If T' is an extension of T and L, L' are the induced logics, then $\beta : L'W \to WL$ exists. If, moreover, T' is the induced extension (posetification) of T, then β is an isomorphism. Furthermore, in the same way as the induced logic L can be seen as the logic of all predicate liftings of T, the induced logic L' is the logic of all monotone predicate of T. These results depend crucially on the fact that the posetification T' of T is defined as a completion with respect to Pos-enriched colimits. On the algebraic side the move to Pos-enriched colimits corresponds to an extension of the presentation results of [24] to functors on (finitary) varieties enriched over posets. In particular, a functor $L' : \mathsf{DL} \to \mathsf{DL}$ preserves enriched sifted colimits if and only if it is definable by *monotone* operations and equations. To see the relevance of a presentation result specific to monotone operations, observe that in the example of positive modal logic it is indeed the case that both $\Box$ and $\Diamond$ are monotone.

2 On coalgebras and coalgebraic logic

I. Coalgebras. A Kripke model (W, R, v) with $R \subseteq W \times W$ and $v : W \to 2^{\mathsf{AtProp}}$ can also be described as a coalgebra $W \to \mathcal{P}W \times 2^{\mathsf{AtProp}}$, where $\mathcal{P}W$ stands for the powerset of W. This point of view suggests to generalize modal logic from Kripke frames to coalgebras

$$\xi : X \to TX$$

where T may now be any functor $T : \mathsf{Set} \to \mathsf{Set}$. We obtain back Kripke models by putting $TX = \mathcal{P}X \times 2^{\mathsf{AtProp}}$. We also get the so-called bounded morphisms or p-morphisms as coalgebras morphisms, that is, as maps $f : X \to X'$ such that $Tf \circ \xi = \xi' \circ f$.

II. Coalgebras and algebras. More generally, for any category $\mathcal{C}$ and functor $T : \mathcal{C} \to \mathcal{C}$, we have the category $\mathsf{Coalg}(T)$ of T-coalgebras with objects and morphisms as above. Dually, $\mathsf{Alg}(T)$ is the category where the objects $TX \xrightarrow{\alpha} X$ are arrows in $\mathcal{C}$ and where the morphisms $f : (X, \alpha) \to (X, \alpha')$ are arrows $f : X \to X'$ in $\mathcal{C}$ such that $f \circ \alpha = \alpha' \circ Tf$. It is worth noting that T-coalgebras over $\mathcal{C}$ are dual to T^{op}-algebras over $\mathcal{C}^{op}$.

III. Duality of Boolean algebras and sets. The abstract duality between algebras and coalgebras becomes interesting if we carry it over a concrete duality, such as the dual adjunction between the category Set of sets and functions and the category BA of Boolean algebras. We denote by $P : \mathsf{Set}^{op} \to \mathsf{BA}$ the functor taking powersets and by $S : \mathsf{BA} \to \mathsf{Set}^{op}$ the functor taking ultrafilters. Alternatively, we can describe these functors by $PX = \mathsf{Set}(X, 2)$ and $SA = \mathsf{BA}(A, \mathbb{2})$, which also determines their action on arrows (here $\mathbb{2}$ denotes the two-element Boolean algebra). P and S are adjoint, satisfying $\mathsf{Set}(X, SA) \cong \mathsf{BA}(A, PX)$. Restricting P and S to finite Boolean algebras/sets, this adjunction becomes a dual equivalence.

IV. Boolean logics for coalgebras, syntax. What now are logics for coalgebras? We follow a well-established methodology in modal logic ([7]) and study modal logics via the associated category of modal algebras. More formally, given a modal logic $\mathcal{L}$ extending Boolean propositional logic and with associated category $\mathcal{A}$ of modal algebras, we describe $\mathcal{L}$ by a functor

$$L : \mathsf{BA} \to \mathsf{BA}$$

so that the category $\mathsf{Alg}(L)$ of algebras for the functor L coincides with $\mathcal{A}$. In particular, the Lindenbaum algebra of $\mathcal{L}$ will be the initial L-algebra.

Example 2.1. Let $T = \mathcal{P}$ be the powerset functor and $L : \mathsf{BA} \to \mathsf{BA}$ be the functor mapping an algebra A to the algebra LA generated by $\Box a$, $a \in A$, and quotiented by the relation stipulating that $\Box$ preserves finite meets, that is,

$$\Box\top = \top \qquad \Box(a \wedge b) = \Box a \wedge \Box b \tag{4}$$

Remark 2.2. $\mathsf{Alg}(L)$ is the category of modal algebras (Boolean algebras with operators), a result which appears to be explicitly stated first in [1].

V. Boolean logics for coalgebras, semantics. The semantics of such a logic is described by a natural transformation

$$\delta : LP \to PT^{op}$$

Intuitively, each modal operator in LPX is assigned its meaning as a subset of TX. More formally, δ allows us to lift $P : \mathsf{Set}^{op} \to \mathsf{BA}$ to a functor $P^\sharp : \mathsf{Coalg}(T) \to \mathsf{Alg}(L)$, and if we take a formula ϕ to be an element of the initial L-algebra (the Lindenbaum algebra of the logic), then the semantics of ϕ as a subset of a coalgebra (X, ξ) is given by the unique arrow from that initial algebra to $P^\sharp(X, \xi)$.

Example 2.3. We define the semantics $\delta_X : LPX \to P\mathcal{P}^{op}X$ by, for $a \in PX$,

$$\Box a \mapsto \{b \in \mathcal{P}X \mid b \subseteq a\}. \tag{5}$$

Remark 2.4. It is an old result in domain theory that δ_X is an isomorphism for finite X ([1]). This implies completeness of axioms (4) with respect to Kripke semantics.

VI. Functors having presentations by operations and equations. One might ask when a functor $L : \mathsf{BA} \to \mathsf{BA}$ can legitimately be considered to give rise to a modal logic. For us, in this paper, a minimal requirement on L is that $\mathsf{Alg}(L)$ is a variety in the sense of universal algebra, that is, that $\mathsf{Alg}(L)$ can be described by operations and equations, the operations then corresponding to modal operators and the equations to axioms. This happens if L is determined by its action on finitely generated free algebras (see [24]). These functors are also characterized as functors having presentations by operations and equations, or as functors preserving sifted colimits. Most succinctly, they are precisely those functors that arise as left Kan-extensions of the inclusion functor of the full subcategory of BA consisting of free algebras on finitely many generators.

VII. The (finitary, Boolean) coalgebraic logic of a Set*-functor.* The general considerations laid out above suggest to define the finitary (Boolean) coalgebraic logic associated to a given functor $T : \mathsf{Set} \to \mathsf{Set}$ as

$$\mathbf{L}Fn = PT^{op}SFn \tag{6}$$

where Fn denotes the free Boolean algebra over n generators, for n ranging over natural numbers. The semantics $\boldsymbol{\delta}$ is given by observing that natural transformations $\boldsymbol{\delta} : \mathbf{L}P \to PT$ are in bijection with natural transformations

$$\hat{\boldsymbol{\delta}} : \mathbf{L} \to PT^{op}S \tag{7}$$

so that we can define $\hat{\boldsymbol{\delta}}$ to be the identity on finitely generated free algebras.

More explicitly, $\mathbf{L}A$ can be represented as the free BA over $\{\sigma(a_1, \dots a_n) \mid \sigma \in PT^{op}SFn, a_i \in A, n < \omega\}$ modulo appropriate axioms, with $\boldsymbol{\delta}_X : \mathbf{L}PX \to PT^{op}X$ given by $\boldsymbol{\delta}\sigma(a_1, \dots a_n) = PT^{op}(\hat{a})(\sigma)$ where $\hat{a} : X \to SFn$ is the adjoint transpose of $(a_1, \dots a_n) : n \to UPX$, with the forgetful functor $U : \mathsf{BA} \to \mathsf{Set}$ being right adjoint of F. [4] Of course, in concrete examples one is often able to obtain much more succinct presentations:

Proposition 2.5. *With $T = \mathcal{P}$, the functor $\mathbf{L}$ defined by* (6) *is isomorphic to the functor L of Example 2.1.*

Proof. We need to find a natural isomorphism $\tau : L \to \mathbf{L}$ such that $\boldsymbol{\delta}\tau P = \delta$. Since δ_X is an isomorphism on finite X and $T = \mathcal{P}$ preserves finiteness, the transpose $\hat{\delta}_A : LA \to PT^{op}SA$ is an isomorphism on finite A. This induces an isomorphism $\tau Fn : LFn \to PT^{op}SFn = \mathbf{L}Fn$. Since L is defined by a presentation, we know from [24] that L is determined by its restriction to free finitely generated Boolean algebras. Hence τ extends to the required isomorphism.

VIII. Positive coalgebraic logic. It is evident that, at least for some of the developments above, not only the functor T, but also the categories Set and BA can be considered parameters. Accordingly, one expects that positive coalgebraic logic takes place over the category DL of (bounded) distributive lattices which in turn, is part of an adjunction $P' : \mathsf{Pos}^{op} \to \mathsf{DL}$, taking upsets, and $S' : \mathsf{DL} \to \mathsf{Pos}^{op}$, taking prime filters, or, in other words, $P'X = \mathsf{Pos}(X, \mathbb{2})$ and $S'A = \mathsf{DL}(A, \mathbb{2})$ where $\mathbb{2}$ is, as before, the two-chain (possibly considered as a distributive lattice). Consequently, the 'natural semantics' of positive logics is 'ordered Kripke frames'. That is, we may define a logic for T'-coalgebras, with $T' : \mathsf{Pos} \to \mathsf{Pos}$, to be given by a natural transformation

$$\boldsymbol{\delta}' : \mathbf{L}'P' \to P'T'^{op} \tag{8}$$

where $\mathbf{L}'$ is a functor required to be determined by finitely generated free distributive lattices and $\boldsymbol{\delta}'$ is given by its transpose in the same way as (7).

Example 2.6. Let T' be the convex powerset functor $\mathcal{P}'$ and $L' : \mathsf{DL} \to \mathsf{DL}$ be the functor mapping a distributive lattice A to the distributive lattice $L'A$ generated by $\Box a$ and $\Diamond a$ for all $a \in A$, and quotiented by the relations stipulating that $\Box$ preserves finite meets, $\Diamond$ preserves finite joins, and

$$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b) \qquad \Box(a \vee b) \leq \Diamond a \vee \Box b \tag{9}$$

The natural transformation $\delta'_X : L'P'X \to P'\mathcal{P}'^{op}X$ is defined by, for $a \in P'X$,

$$\Diamond a \mapsto \{b \in \mathcal{P}X \mid b \cap a \neq \emptyset\}, \tag{10}$$

the clause for $\Box a$ being the same as in (5).

[4] Since elements in $PTSFn$ are in one-to-one correspondence with natural transformations $\mathsf{Set}(-, 2^n) \to \mathsf{Set}(T-, 2)$, also knwon as predicate liftings [32], we see that the logic L coincides with the logic of all predicate liftings of [34], with the difference that L also incorporates axioms. The axioms are important to us as otherwise the natural transformation β mentioned in the introduction might not exist.

Remark 2.7. $\mathsf{Alg}(L')$ is the category of positive modal algebras of Dunn [11] and we will show that it is isomorphic to $\mathsf{Alg}(\mathbf{L}')$ in Corollary 3.6. Again we have that for finite X, δ'_X is an isomorphism, a representation first stated in [14,15], the connection with modal logic being given by [36,33,1] and investigated from a coalgebraic point of view in [31].

3 On Pos and Pos-enriched categories

I. The category Pos *of posets and monotone maps.* Pos is complete and cocomplete (even locally finitely presentable [4]), limits being computed as in Set, while for colimits one has to quotient the corresponding colimits obtained in the category of preordered sets and monotone maps (however, directed colimits are computed as in Set, see [4]).

Pos is also cartesian closed, with the internal hom $[X, Y]$ being the poset of monotone maps from X to Y, ordered pointwise.

This paper will consider categories *enriched* in Pos because this automatically takes care of the algebraic operations being monotone. Therefore when we say category, functor, natural transformation in what follows, we always mean the enriched concept. When we want to deal with non-enriched concepts, we always call them *ordinary*. Thus, for example, the category Pos has its underlying ordinary category Pos_o. Everything below with the subscript o is the underlying ordinary thing of the Pos-enriched thing. In particular, we consider Set as discretely enriched over Pos. Then $D : \mathsf{Set} \to \mathsf{Pos}$, the discrete functor, is trivially Pos-enriched. There are two more Pos-categories appearing in this paper, namely BA and DL. The first one is considered discretely enriched, while in DL the enrichment is a consequence of the natural order induced by operations.

Notice that Pos is actually locally finitely presentable as a symmetric monoidal closed category ([19]), as the ordinary category Pos_o is locally finitely presentable, the one-element poset is a finitely presentable object of Pos_o, and the product $X \times Y$ is a finitely presentable object in Pos_o, whenever both X and Y are. All three Pos-categories mentioned above are locally finitely presentable, the finitely presentable objects being the finite ones.

II. Sifted weights and sifted (co)limits. In order to properly describe Pos-functors and their logics by presentations and axioms, we shall need a detour into the world of ordered varieties. We shall be brief and refer to [27], [9], [29], [20] for more details.

A weight $W : J^{op} \to \mathsf{Pos}$ is called *sifted* if $\Pi_n : [n, \mathsf{Pos}] \to \mathsf{Pos}$ preserves all W-weighted colimits, where Π_n is the functor taking the product of an n-tuple of posets, for every finite discrete poset n. A sifted colimit is a colimit weighted by a sifted weight. Examples of sifted colimits are filtered colimits and reflexive coequalizers. There is one more important example to mention, called codescent object of reflexive coherence datum, an enriched analogue of a reflexive coequalizer ([9,28]). In particular, any poset p can be canonically expressed as such a colimit (in the same way an internal category can be specified by its object of objects, its object of arrows and object of composable pairs with the corresponding morphisms between them).

Codescent objects in can be obtained by means of coinserters and coequifiers ([28]); but as the latter are trivial in Pos and Pos-categories, constructing codescent objects reduces to coinserters. Similar considerations apply for the dual limit notion which arises simply as an inserter in Pos.

III. Functors preserving sifted colimits and their equational presentation. Denote by Set_f the category of finite sets and maps and by ι the composite $\mathsf{Set}_f \hookrightarrow \mathsf{Set} \xrightarrow{D} \mathsf{Pos}$. Then Pos is the free cocompletion of Set_f under (enriched) sifted colimits. Equivalently, it is the free cocompletion of Set_f under filtered colimits and codescent objects of reflexive coherence data [27].

A functor $\mathcal{T} : \mathsf{Pos} \to \mathsf{Pos}$ is called *strongly finitary* if one of the three equivalent conditions below holds: (i) $\mathcal{T}$ is isomorphic to the left Kan extension along ι of its restriction, that is $\mathcal{T} \cong \mathsf{Lan}_\iota(\mathcal{T}\iota)$; (ii) $\mathcal{T}$ preserves filtered colimits and codescent objects of reflexive coherence data; (iii) $\mathcal{T}$ preserves sifted colimits.

Recall there are monadic (enriched) adjunctions $F \dashv U : \mathsf{BA} \to \mathsf{Set}$, $F' \dashv U' : \mathsf{DL} \to \mathsf{Pos}$, where U and U' are the corresponding forgetful functors. We denote by $\mathbf{J} : \mathsf{BA}_{\mathit{ff}} \to \mathsf{BA}$ and $\mathbf{J}' : \mathsf{DL}_{\mathit{ff}} \to \mathsf{DL}$ the inclusion functors of the full subcategories spanned by the algebras which are free on finite (discrete po)sets.

Lemma 3.1. *$\mathbf{J}$ and $\mathbf{J}'$ exhibit BA, respectively DL, as the free cocompletions under sifted colimits of $\mathsf{BA}_{\mathit{ff}}$ and $\mathsf{DL}_{\mathit{ff}}$. In particular, these functors are dense.*

Proof. We know that the ordinary functor $\mathbf{J}_o : (\mathsf{BA}_{\mathit{ff}})_o \to \mathsf{BA}_o$ exhibits BA_o as a free cocompletion under ordinary sifted colimits. Now the result for $\mathbf{J}$ follows by noticing that codescent objects of reflexive coherence data are computed as reflexive coequalizers.

For distributive lattices, the result is an instance of Theorem 6.10 of [27], since DL is a finitary variety of ordered algebras (thus, DL is isomorphic to the category of algebras for a strongly finitary monad on Pos).

Corollary 3.2. *A functor $L : \mathsf{BA} \to \mathsf{BA}$ has the form $\mathrm{Lan}_{\mathbf{J}}(L\mathbf{J})$ iff it preserves (ordinary) sifted colimits. A functor $L' : \mathsf{DL} \to \mathsf{DL}$ has the form $\mathrm{Lan}_{\mathbf{J}'}(L'\mathbf{J}')$ iff it preserves sifted colimits.*

Theorem 3.3. *Suppose $L : \mathsf{BA} \to \mathsf{BA}$ and $L' : \mathsf{DL} \to \mathsf{DL}$ preserve sifted colimits. Then they both have an equational presentation.*

Remark 3.4. If we denote by $|\mathsf{Set}_f|$ the skeleton of the category of finite sets, then $[|\mathsf{Set}_f|, \mathsf{Pos}]$ can be seen as the *category of strongly finitary signatures*. The (proof of the) above theorem shows that every functor $L' : \mathsf{DL}_{\mathit{ff}} \to \mathsf{DL}$ (i.e., every L' preserving sifted colimits) has a presentation in the form of a coequalizer

$$\widehat{H_\Gamma} \rightrightarrows \widehat{H_\Sigma} \longrightarrow L'$$

for some strongly finitary signatures Γ and Σ. Here, $\widehat{H_\Sigma}$ is defined as follows: given $\Sigma : |\mathsf{Set}_f| \to \mathsf{Pos}$, $H_\Sigma : \mathsf{Set}_f \to \mathsf{Pos}$ is the *polynomial strongly finitary functor*

$$H_\Sigma n = \coprod_k \mathsf{Set}_f(k,n) \bullet \Sigma k$$

and it extends to a strongly finitary $H_\Sigma : \mathsf{Pos} \to \mathsf{Pos}$ by sifted colimits.

The resulting $\widehat{H_\Sigma} : \mathsf{DL}_{\mathit{ff}} \to \mathsf{DL}$ is given, at a free distributive lattice of the form $F'Dn$, by

$$\widehat{H_\Sigma}(F'Dn) = F'H_\Sigma U'(F'Dn)$$

(see Remark 3.16 of [26]) and, again, it is extended to an endofunctor on DL by means of sifted colimits.

We define a functor $\mathsf{DL} \to \mathsf{DL}$ to have a presentation by monotone operations and equations if it has a presentation by operations and equations in the sense of [24], such that, moreover, all operations are monotone.

Corollary 3.5. *A functor $L' : \mathsf{DL} \to \mathsf{DL}$ has a presentation by monotone operations and equations if and only if L' is the* Pos*-enriched left Kan extension of its restriction to finitely generated free distributive lattices.*

As in Proposition 2.5, we now obtain that

Corollary 3.6. *If T' is the the convex powerset functor, then the functor L' of Example 2.6 is isomorphic to the sifted-colimits preserving functor $\mathbf{L}'$ whose restriction to $\mathsf{DL}_{\mathit{ff}}$ is $P'T'^{op}S'$ as in* (8).

IV. The Pos*-extension of a* Set*-functor.* In order to relate Set and Pos-functors, we recall from [6] the following

Definition 3.7. *Let T be an endofunctor on* Set. *A* Pos*-endofunctor T' is said to be a* Pos-extension *of T if it is locally monotone and if the square*

$$\begin{array}{ccc} \mathsf{Pos} & \xrightarrow{T'} & \mathsf{Pos} \\ {\scriptstyle D}\uparrow & \nwarrow\alpha & \uparrow{\scriptstyle D} \\ \mathsf{Set} & \xrightarrow[T]{} & \mathsf{Set} \end{array} \tag{11}$$

commutes up to an isomorphism $\alpha : DT \to T'D$.

A Pos*-extension T' is called the* posetification *of T if the above square exhibits T' as $\mathsf{Lan}_D DT$ (in the* Pos*-enriched sense), having α as its unit.*

If T is finitary, then its posetification does exist. This can be seen by expressing $\mathsf{Lan}_D(DT)$ as a coend

$$\mathsf{Lan}_D(DT)X = \int^{S\in\mathsf{Set}} [DS, X] \bullet DTS \tag{12}$$

and taking into account that T is determined by its action on finite sets: explicitly, the coend becomes

$$\mathsf{Lan}_D(DT)X = \int^{n \in \mathsf{Set}_f} [Dn, X] \bullet DTn \tag{13}$$

which in turn is the following Pos-coequalizer

$$\coprod_{m,n<\omega} \mathsf{Set}(m,n) \times Tm \times [Dn, X] \rightrightarrows \coprod_{n<\omega} Tn \times [Dn, X] \xrightarrow{\pi} \mathsf{Lan}_D(DT)X \tag{14}$$

Example 3.8. 1. Let $T = \mathrm{Id}$ on Set. Then the discrete connected components functor and the upper-sets-functor are both extensions of T, while $\mathrm{Id} : \mathsf{Pos} \to \mathsf{Pos}$ is the posetification (recall that the discrete functor D is dense, see [10]).
2. If we take $T = \mathcal{P}_f$ to be the (finite) power-set functor, then its posetification is the (finitely generated) convex power-set functor, with the Egli-Milner order ([26,6]).

Posetifications of (finitary) Set-functors are immediate examples of strongly finitary Pos-functors. Briefly, one can say that a Pos-functor is a posetification if it has a presentation by monotone operations and discrete arities. In fact, we can be much more precise: a functor $T' : \mathsf{Pos} \to \mathsf{Pos}$ is the posetification of a finitary Set-functor if it is strongly finitary and preserves discrete sets.

V. Morphisms of logical connections. We recall the (enriched) logical connections (dual adjunctions, see [25]) between sets and Boolean algebras, and between posets and distributive lattices. Both should be seen as Pos-enriched, where for the first logical connection the enrichment is discrete. They are related as follows:

$$\begin{array}{ccc} \mathsf{Set}^{op} & \underset{P}{\overset{S}{\leftrightarrows}} \ (\perp) & \mathsf{BA} \\ {\scriptstyle D^{op}} \downarrow & & \downarrow {\scriptstyle W} \\ \mathsf{Pos}^{op} & \underset{P'}{\overset{S'}{\leftrightarrows}} \ (\perp) & \mathsf{DL} \end{array} \tag{15}$$

In the top row of the above diagram, recall again that P is the contravariant powerset functor, while S maps a Boolean algebra to its set of ultrafilters. The bottom row has P' mapping a poset to the distributive lattice of its upper-sets, and S' associating to each distributive lattice the poset of its prime filters. About the pair of functors connecting the two logical connections: D was introduced earlier as the discrete functor, while W is the functor associating to each Boolean algebra its underlying distributive lattice.

It is easy to see that the pair (D^{op}, W) is a *morphism of adjunctions* in the sense of [30]. This means that the equalities

$$P'D^{op} = WP, \quad D^{op}S = S'W, \quad \epsilon' D^{op} = D^{op}\epsilon \tag{16}$$

hold, where ϵ and ϵ' are the counits of $S \dashv P$ and $S' \dashv P'$, respectively.

4 Positive coalgebraic logic

Consider T a Set-endofunctor and T' an extension of T to Pos as in (11). Logics for the pair (T, T') are given by functors $L : \mathsf{BA} \to \mathsf{BA}$ and $L' : \mathsf{DL} \to \mathsf{DL}$ and natural transformations

$$\delta : LP \to PT^{op} \qquad \delta' : L'P' \to P'{T'}^{op}$$

assigning to syntax as given by presentations of L and L' the corresponding semantics in subsets or upper sets. To compare L and L' we need the isomorphism $\alpha : DT \to T'D$ saying that T' extends T, and also the relation $WP = P'D$ from (16) (which formalizes the trivial observation that taking upsets of a discrete set is the same as taking all subsets). Referring back to the introduction, we now make the following

Definition 4.1. *We say that a logic* (L', δ') *for* T' *is* a positive fragment *of the logic* (L, δ) *for* T*, if there is a natural transformation* $\beta : L'W \to WL$ *with* $W\delta \circ \beta P = P'\alpha^{op} \circ \delta' D^{op}$*, or, in diagrams*

$$\begin{array}{ccccc} \mathsf{Set}^{op} & \xrightarrow{P} & \mathsf{BA} & \xrightarrow{W} & \mathsf{DL} \\ {\scriptstyle T^{op}}\downarrow & \swarrow\delta & {\scriptstyle L}\downarrow & \swarrow\beta & \downarrow{\scriptstyle L'} \\ \mathsf{Set}^{op} & \xrightarrow[P]{} & \mathsf{BA} & \xrightarrow[W]{} & \mathsf{DL} \end{array} \quad = \quad \begin{array}{ccccc} \mathsf{Set}^{op} & \xrightarrow{D^{op}} & \mathsf{Pos}^{op} & \xrightarrow{P'} & \mathsf{DL} \\ {\scriptstyle T^{op}}\downarrow & \swarrow\alpha^{op} & {\scriptstyle T'^{op}}\downarrow & \swarrow\delta' & \downarrow{\scriptstyle L'} \\ \mathsf{Set}^{op} & \xrightarrow[D^{op}]{} & \mathsf{Pos}^{op} & \xrightarrow[P']{} & \mathsf{DL} \end{array} \tag{17}$$

We call (L', δ') the (maximal) positive fragment *of* (L, δ) *if* β *is an isomorphism.*

Recall that we defined the logics $\mathbf{L}, \mathbf{L}'$ induced by T and an extension T' as $\mathbf{L} = PTS$ and $\mathbf{L}' = P'{T'}^{op}S'$ on finitely generated free objects. Our desired result is to prove that a certain canonically given $\beta : \mathbf{L}'W \to W\mathbf{L}$ is an isomorphism. The difficulty, as well as the need for the proviso that T preserves weak pullbacks, stems from the fact that in DL (as opposed to BA) the class of functors determined on finitely generated free algebras is strictly smaller than the class of functors determined on finitely presentable (=finite) algebras. As stepping stones, therefore, we first investigate what happens in the cases where the functors L, L' are determined on all algebras and on finitely presentable algebras, before we turn the situation of functors determined on strongly finitely presentable (=finitely generated free algebras).

I. The case of $L' = P'{T'}^{op}S'$ *on all algebras.* We shall associate to any extension $\alpha : DT \to T'D$ the pairs (L, δ) and (L', δ') corresponding to T and T' respectively, with $L = PT^{op}S$ and $\delta = PT^{op}\epsilon : PT^{op}SP \to PT^{op}$, $L' = P'{T'}^{op}S'$ and δ' being defined analogously. Now the following is a consequence of (D^{op}, W) being a morphism of adjunctions (see (16)). We then immediately obtain an isomorphism β:

Proposition 4.2. *Given an extension $\alpha : DT \to T'D$, the isomorphism*

$$\begin{array}{ccccccc}
 & & & \overset{L}{\curvearrowright} & & & \\
\mathsf{BA} & \xrightarrow{S} & \mathsf{Set}^{op} & \xrightarrow{T^{op}} & \mathsf{Set}^{op} & \xrightarrow{P} & \mathsf{BA} \\
{\scriptstyle W}\downarrow & & {\scriptstyle D^{op}}\downarrow & \nearrow\alpha^{op} & \downarrow{\scriptstyle D^{op}} & & \downarrow{\scriptstyle W} \\
\mathsf{DL} & \xrightarrow[S']{} & \mathsf{Pos}^{op} & \xrightarrow[T'^{op}]{} & \mathsf{Pos}^{op} & \xrightarrow[P']{} & \mathsf{DL} \\
 & & & \underset{L'}{\curvearrowright} & & &
\end{array}$$

exhibits $L' = P'{T'}^{op}S'$ as the maximal positive fragment of $L = PT^{op}S$.

II. The case of $L' = P'T'^{op}S'$ on finitely presentable algebras. A similar result holds if we define logics via $PT^{op}SA$ for finitely presentable A, as we are going to show now. To this end, we use the subscript $(-)_f$ to denote the restriction to finite[5] objects as e. g. when writing the dense inclusions $I : \mathsf{Set}_f \to \mathsf{Set}$, $I' : \mathsf{Pos}_f \to \mathsf{Pos}$, $J : \mathsf{BA}_f \to \mathsf{BA}$ and $J' : \mathsf{DL}_f \to \mathsf{DL}$. Note that we have the following commuting diagram

$$\begin{array}{ccc}
S_f \dashv P_f & \xrightarrow{(D_f^{op}, W_f)} & S'_f \dashv P'_f \\
{\scriptstyle (I^{op},J)}\downarrow & & \downarrow{\scriptstyle (I'^{op},J')} \\
S \dashv P & \xrightarrow[(D^{op},W)]{} & S' \dashv P'
\end{array} \tag{18}$$

in the category of transformations of adjoints.

Define $(\bar{L}, \bar{\delta})$ for T as $\bar{L} = \mathsf{Lan}_J(LJ)$ and $\bar{\delta} = \bar{L}P \overset{\gamma P}{\to} LP \overset{\delta}{\to} PT^{op}$, with $\gamma : \bar{L} = \mathrm{Lan}_J(LJ) \to L$ induced by the the left Kan extension. By construction, $\bar{L}$ is finitary and is given by $PT^{op}S$ on finite(ly presentable) Boolean algebras. Similarly, obtain $(\bar{L}', \bar{\delta}')$ for T'.

Since W is left adjoint,[6] $\mathrm{Lan}_J(LJ)$ is preserved by W. Thus, to define an (iso)morphism $\bar{\beta} : \bar{L}'W = \mathsf{Lan}_{J'}(L'J')W \to W\bar{L} = \mathsf{Lan}_J(LJ)W$, it suffices to take the restriction along J of the isomorphism of Proposition 4.2, namely $\bar{\beta}_f : L'J'W_f = P'{T'}^{op}S'J'W_f \cong WPT^{op}SJ = WL$.

Proposition 4.3. *The isomorphism $\bar{\beta}$ exhibits $(\bar{L}, \bar{\delta})$ as the maximal positive fragment of (L, δ).*

Remark 4.4. Note that the above proposition does still not give us the desired result, as $\bar{L}'$ is not necessarily uniquely determined by its action on finitely

[5] As Pos is locally finitely presentable as closed category, and ordinary categories $\mathsf{Set}_o, \mathsf{DL}_o, \mathsf{BA}_o$ are also locally finitely presentable, it follows that the finitely presentable objects in all the above categories are precisely the same as in the ordinary case, i.e. the ones for which the underlying set is finite.

[6] The (enriched) right adjoint of W sends a distributive lattice A to the Boolean algebra of complemented elements in A (also known as the center of A).

generated free algebras and, therefore, need not give rise to a variety of modal algebras.[7]

III. The case of $\mathbf{L}' = L' = P'T'^{op}S'$ *on finitely generated algebras.*

Definition 4.5. *Let* T' *be a* Pos*-endofunctor. We define* strongly finitary logic *for* T' *to be the pair* $(\mathbf{L}', \boldsymbol{\delta}')$*, where:*

- $\mathbf{L}' : \mathsf{DL} \to \mathsf{DL}$ *is a* Pos*-functor preserving sifted colimits, whose restriction to free finitely generated distributive lattices is* $L'\mathbf{J}' = P'T'^{op}S'\mathbf{J}'$*. More precisely,* $\mathbf{L}' = \mathsf{Lan}_{\mathbf{J}'}(L'\mathbf{J}')$.
- $\boldsymbol{\delta}' : \mathbf{L}'P' \to P'T'^{op}$ *is the composite* $\boldsymbol{\delta}' = P'T'^{op}\varepsilon' \cdot \boldsymbol{\gamma}'P'$*, where* $\boldsymbol{\gamma}' : \mathbf{L}' = \mathsf{Lan}_{\mathbf{J}'}(L'\mathbf{J}') \to L'$ *is the natural transformation corresponding to the Kan extension.*

Remark 4.6. By the above definition, $\mathbf{L}'$ preserves sifted colimits. Thus it has an equational presentation; this is precisely what we required for logic functors.

Theorem 4.7. *Let* T *be a* Set*-endofunctor and* T' *a* Pos*-extension of* T *which* preserves coreflexive inserters. *Then* $(\bar{L}', \bar{\delta}')$ *and* $(\mathbf{L}', \boldsymbol{\delta}')$ *coincide. In particular, it follows that* $\mathbf{L}'$ *is* the maximal positive fragment *of* $\mathbf{L}$.

Remark 4.8. The isomorphism $(\bar{L}, \bar{\delta}) \cong (\mathbf{L}, \boldsymbol{\delta})$ of the corresponding Boolean logic for Set-functors was established in [24]. (Recall that $\mathbf{L}$ was introduced in (6), while $\bar{L}$ appeared in Paragraph II. above.)

Proposition 4.9 ([6]). *Let* T *be any finitary* Set*-functor and* T' *its posetification. Then* T *preserves weak pullbacks if and only if* T' *preserves exact squares.*

Proposition 4.10. *If* T' *is a* Pos*-endofunctor (thus locally monotone) which preserves exact squares, then it preserves embeddings and coreflexive inserters.*

The reader should think of an exact square as being the Pos-enriched analogue of a weak pullback (see [13], [5] or [6] for the precise definition).

As a consequence of all the results of this section, we obtain

Theorem 4.11. *Let* $T : \mathsf{Set} \to \mathsf{Set}$ *be a finitary weak-pullback preserving functor and* $T' : \mathsf{Pos} \to \mathsf{Pos}$ *its posetification. Let* $(\mathbf{L}, \boldsymbol{\delta})$ *and* $(\mathbf{L}', \boldsymbol{\delta}')$ *be the associated logics of* T *and* T'*, respectively. Then* $(\mathbf{L}', \boldsymbol{\delta}')$ *is the maximal positive fragment of* $(\mathbf{L}, \boldsymbol{\delta})$.

Example 4.12. For $T = \mathrm{Id}$, the corresponding finitary logics is $\mathbf{L} = \mathrm{Id}$ on BA, with trivial semantics $\boldsymbol{\delta} : LP \to PT^{op}$. It allows the extension $T' = DC$, the discrete connected components functor. Notice that T' does not preserve embeddings, neither coreflexive inserters. The corresponding logic $\mathbf{L}'$ is given by the constant functor to the distributive lattice $\mathbb{2}$. Thus $\beta : \mathbf{L}'W \to W\mathbf{L}$ fails to be an isomorphism (it is just the unique morphism from the initial object).

[7] However, this does hold for $\bar{L}$, see [24].

Our introductory example of positive modal logic is now regained as an instance of this theorem.[8] It can also easily be adapted to Kripke polynomial functors. More interesting is the case of the probability distribution functor. We know from the theorem above that it has a maximal positive fragment, but an explicit description still needs to be worked out.

5 Monotone predicate liftings

In this section we show that the logic of the posetification T' of T coincides with the logic of all monotone predicate liftings of T.

Recall that a predicate lifting [32,34] of arity n for T is an ordinary natural transformation $\heartsuit : \mathsf{Set}_o(-, 2^n) \to \mathsf{Set}_o(T-, 2)$,[9] or, using the ordinary adjunction $D_o \dashv V : \mathsf{Pos}_o \to \mathsf{Set}$, an ordinary natural transformation

$$\heartsuit : \mathsf{Pos}_o(D_o-, [n, \mathbb{2}]) \to \mathsf{Pos}_o(D_oT-, \mathbb{2})$$

It is called *monotone* if it lifts to a natural transformation

$$\heartsuit : \mathsf{Pos}(D-, [Dn, \mathbb{2}]) \to \mathsf{Pos}(DT-, \mathbb{2})$$

By identifying a predicate lifting with an map $\heartsuit : T(2^n) \to 2$, the above says that $\heartsuit$ is monotone if for all $\overline{a_1} \leq \overline{a_2} : D_oX \to [D_on, \mathbb{2}]$, we have that $\overline{\heartsuit \circ Ta_1} \leq \overline{\heartsuit \circ Ta_2}$, where $\overline{f} : D_oX \to Y$ denotes the adjoint transpose of $f : X \to VY$.

Consider now a Pos-functor T' (locally monotone!) and a finite poset p. By mimicking the above, we define a predicate lifting for T' of arity p as being a natural transformation[10]

$$\heartsuit : \mathsf{Pos}(-, [p, \mathbb{2}]) \to \mathsf{Pos}(T'-, \mathbb{2})$$

Proposition 5.1. *Let T be a Set-functor and $T' : \mathsf{Pos} \to \mathsf{Pos}$ an extension. Then:*

1. *There is an injection from the set of predicate liftings of T' of arity p into the set of monotone predicate liftings of T of arity Vp.*
   In particular, the set of predicate liftings of T' of discrete arity n embeds into the monotone predicate liftings of T.
2. *In case T' is the posetification of T, the above mapping is a bijection.*

As a corollary, we obtain

[8] A minor issue here is that modal logic usually takes as semantics coalgebras for the (non-finitary) powerset, whereas for the posetification to exist we sofar assumed T to be finitary. There are two solutions to this. One is to note that going from T to its finitary coreflection T_ω and then to its posetification T'_ω does not change the functors $\mathbf{L}, \mathbf{L}'$ on the algebraic side. The second is to prove that the posetification exists despite the functor not being accessible.

[9] Equivalently, it can be described as an element $\heartsuit \in \mathsf{Set}(T(2^n), 2)$.

[10] Which can be identified with $\heartsuit \in \mathsf{Pos}(T'([p, \mathbb{2}]), \mathbb{2})$.

Corollary 5.2. *Let T be a finitary* Set*-functor. If the posetification T' of T preserves embeddings, then the logic of all monotone predicate liftings of T is expressive.*

Proof. The final T'-coalgebra and the final T-coalgebra coincide ([6]). If T' preserves embeddings and is finitary, then the logic of all predicate liftings of finite discrete arity of T' is expressive for the final T'-coalgebra ([17]) and therefore also for the final T-coalgebra. By the above proposition, all monotone predicate liftings are also expressive for T-coalgebras.

Remark 5.3. We know from [6] that if T preserves weak pullbacks then T' preserves embeddings. So the above theorem applies to weak-pullback preserving functors. This result was obtained in [23, Cor 6.9] already in a different way.

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A Proof of Thm. 3.3

For L it is well-known. For L', proceed as follows:

1. Observe that $[F'_f, U'] : [\mathsf{DL}_{ff}, \mathsf{DL}] \to [\mathsf{Set}_{fp}, \mathsf{Pos}]$, sending a functor $L' : \mathsf{DL}_{ff} \to \mathsf{DL}$ to the composite $U'L'F'_f$, is of descent type. This follows immediately from Lemma 3.14 in [26]. Above, $F'_f : \mathsf{Set}_f \to \mathsf{DL}_{ff}$ is the domain-codomain restriction of $F'D : \mathsf{Set} \to \mathsf{Pos} \to \mathsf{DL}$.
2. The functor $[E, -] : [\mathsf{Set}_f, \mathsf{Pos}] \to [|\mathsf{Set}_f|, \mathsf{Pos}]$, where $|\mathsf{Set}_f|$ is the skeleton of the category of finite sets and $E : |\mathsf{Set}_f| \to \mathsf{Set}_f$ is the inclusion, is monadic. (Again, this follows from Lemma 3.14 of [26]).
3. The composite

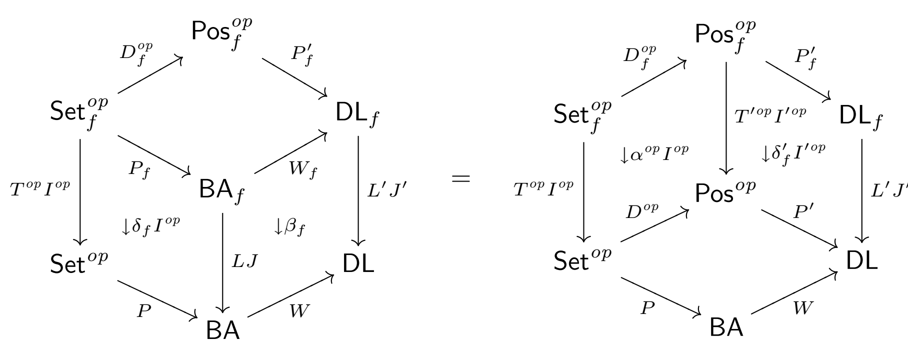
$$[\mathsf{DL}_{ff}, \mathsf{DL}] \xrightarrow{[F'_f, U']} [\mathsf{Set}_f, \mathsf{Pos}] \xrightarrow{[E,-]} [|\mathsf{Set}_f|, \mathsf{Pos}]$$

   is of descent type. This follows from Theorem 3.18 of [26].

□

B Proof of Prop. 4.3

The easiest way to prove this is to show that $\bar{\beta}_f$, defined by the above, fulfills



(19)

But this follows from Proposition 4.2 and (18). □

C Proof of Thm. 4.7

Proof. In order to show $(\bar{L}', \bar{\delta}') = (\mathbf{L}', \boldsymbol{\delta}')$, it is enough to check that $\bar{L}'$ and $\mathbf{L}'$ agree on finite distributive lattices, as both are finitary. That is, we need to show that $\mathbf{L}'$ is $P'T'^{op}S'$ on any finite distributive lattice, not just on the free lattices with finitely many discrete generators. In particular, this will also indicate $\bar{\delta}' \cong \boldsymbol{\delta}'$.

(1) Using that the free-distributive lattice monad $U'F' : \mathsf{Pos} \to \mathsf{Pos}$ is strongly finitary, one can exhibit every (finite) distributive lattice as a coinserter of free (finite) ones.[11]

Namely, take A to be a finite distributive lattice and consider the counit $\varepsilon_A : F'U'A \to A$ in DL. It is an so-morphism ([27]), hence a coinserter of some pair $A' \rightrightarrows F'U'A$ (by factoring the pair through its image, we can assume without loss of generality that A' is finite). Now post-compose this pair with $\varepsilon_{A'} : F'U'A' \to A'$ to obtain $F'U'A' \rightrightarrows F'U'A$.

Again, since $\varepsilon_{A'}$ is an so-morphism, it is a coinserter. Hence $F'U'A' \rightrightarrows F'U'A$ and $A' \rightrightarrows F'U'A$ have the same "coinserter cocones". This exhibits A as the coinserter of $F'U'A' \rightrightarrows F'U'A$.

(2) We need to check that $\mathbf{L}'$ and $\bar{L}'$ agree on all free distributive lattices on finite posets.

Given a finite poset p, exhibit it as a coinserter of its reflexive coherence datum

$$p_1 \; \overset{d_0^1}{\underset{d_1^1}{\underset{\xleftarrow{\;i_0^0\;}}{\rightrightarrows}}} \; p_0 \xrightarrow{\quad q \quad} p \tag{20}$$

where p_0 is the discrete poset of elements of p and p_1 is the (discrete po)set of ordered pairs in p. Apply now $L'F'$; we get

$$\begin{array}{ccccc} L'F'p_1 & \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} & L'F'p_0 & \longrightarrow & L'F'p \\ \Vert & & \Vert & & \\ P'T'^{op}[p_1, \mathbb{2}] & & P'T'^{op}[p_0, \mathbb{2}] & & \end{array} \tag{21}$$

The above diagram is colimiting, as F' is left adjoint and $\mathbf{L}'$ preserves sifted colimits by definition. We just need to show that the colimit is in fact $P'T'^{op}S'F'p = P'T'^{op}[p, \mathbb{2}]$. First, use that S' is a left adjoint to move the diagram (20) and its colimit from DL to Pos^{op}.

Thus in Pos, we obtain $[q, \mathbb{2}]$ as the inserter of the pair $[d_0^1, \mathbb{2}], [d_1^1, \mathbb{2}]$. By hypothesis, T' preserves it. Now apply Lemma C.1 and we are done. ☐

Lemma C.1. *The contravariant functor P' maps coreflexive inserters in* Pos *to reflexive coinserters in* DL.

Proof. Notice that $U'P' = [-, \mathbb{2}]$ and that U' is monadic, thus conservative, and preserves sifted colimits (as DL is an ordered variety), in particular reflexive coinserters. Thus it is enough to show that $[-, \mathbb{2}]$ transforms coreflexive inserters into (reflexive) coinserters.

Consider therefore a pair of two monotone maps (c, d) with common right inverse i in Pos

$$X \; \overset{c}{\underset{d}{\underset{\xleftarrow{\;i\;}}{\rightrightarrows}}} \; Y \tag{22}$$

[11] Intuitively, this means that in distributive lattices equations can be expressed by (pairs of) inequations.

and form the inserter of the above data: $\mathsf{Ins} = \{x \in X \mid cx \leq dx\}$ and $\mathsf{Ins} \xrightarrow{e} X$ is the inclusion. In particular, the diagram below is an exact square (see Def. C.2):

$$\begin{array}{ccc} \mathsf{Ins} & \xrightarrow{e} & X \\ {\scriptstyle e}\downarrow & \swarrow & \downarrow{\scriptstyle c} \\ X & \xrightarrow[d]{} & Y \end{array}$$

By Lemma C.4, we obtain $[d, \mathbb{2}] \circ \exists_c = \exists_e \circ [e, \mathbb{2}]$. Here $\exists_e$ denotes the left adjoint of $[e, \mathbb{2}]$ (see Lemma C.3). As both e and c are embeddings (c being a split mono), $[e, \mathbb{2}] \circ \exists_e = \mathrm{id}$ and $[c, \mathbb{2}] \circ \exists_d = \mathrm{id}$ by Lemma C.3.

Thus applying $[-, \mathbb{2}]$ to the diagram (22) augmented by $\mathsf{Ins} \xrightarrow{e} X$ and $\exists_e, \exists_c$, it exhibits $[E, \mathbb{2}]$ as the split coinserter of of $[c, \mathbb{2}], [d, \mathbb{2}]$. Therefore $[-, \mathbb{2}]$ maps coreflexive inserters to (reflexive and split) coinserters. □

Definition C.2 ([13]). *An exact square in* Pos *is a diagram*

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & X \\ {\scriptstyle \beta}\downarrow & \swarrow & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array} \tag{23}$$

with $f\alpha \leq g\beta$, *such that*

$$\forall x \in X, y \in Y.\ f(x) \leq g(y) \ \Rightarrow \exists p \in P.\ x \leq \alpha(p) \wedge \beta(p) \leq y \tag{24}$$

Lemma C.3. *Let* $e : E \to X$ *be an embedding of posets. Then* $[e, \mathbb{2}]$ *has a right inverse.*

Proof. Any poset can be seen as a category enriched over $\mathbb{2}$, and any monotone map $e : (E, \leq) \to (X, \leq)$ as an enriched functor. Pre-composition with e gives a functor between posets $[e, \mathbb{2}] : [X, \mathbb{2}] \to [E, \mathbb{2}]$ which has a left (and a right) adjoint, given by left (and right) Kan extensions. Explicitly, the left adjoint $\exists_e$ maps an upper set ϕ to the up-set closure of its image $e(\phi)\uparrow$.

Remark e is an embedding precisely when it is fully faithful, thus by [18], Prop. 4.23 the unit of the adjunction (the natural transformation corresponding to left Kan extensions) is an isomorphism. Translated into Pos-language, this means equality, thus $[e, \mathbb{2}] \circ \exists_e = \mathrm{id}$. □

Lemma C.4. *The diagram* (23) *exhibits an exact square iff the Beck-Chevalley condition holds, namely* $[g, \mathbb{2}] \circ \exists_f = \exists_\alpha \circ [\beta, \mathbb{2}]$*:*

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & X \\ {\scriptstyle \beta}\downarrow & \swarrow & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} [P, \mathbb{2}] & \xleftarrow{[\alpha, \mathbb{2}]} & [A, \mathbb{2}] \\ {\scriptstyle \exists_\beta}\downarrow & & \downarrow{\scriptstyle \exists_f} \\ [Y, \mathbb{2}] & \xleftarrow[{[g, \mathbb{2}]}]{} & [Z, \mathbb{2}] \end{array}$$

Proof. It follows easily by direct computation. □

D Proof of Prop. 4.10

As each embedding $e : X \to Y$ can be realized as an exact square

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & \swarrow & \downarrow{\scriptstyle e} \\ X & \xrightarrow{e} & Y \end{array}$$

the first assertion follows immediately.

For the second one, consider

$$E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y, \qquad h : Y \to X$$

a (coreflexive) inserter. In particular,

$$\begin{array}{ccc} E & \xrightarrow{e} & X \\ {\scriptstyle e}\downarrow & \swarrow & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

is an exact square, thus mapped by T' into an exact square

$$\begin{array}{ccc} T'E & \xrightarrow{T'e} & T'X \\ {\scriptstyle T'e}\downarrow & \swarrow & \downarrow{\scriptstyle T'f} \\ T'X & \xrightarrow{T'g} & T'Y \end{array}$$

Let $u : U \to T'X$ be a monotone map such that $T'f \circ u \leq T'g \circ u$. For each $x \in U$, there is thus $w \in T'E$ with $u(x) = T'E(w)$. As $T'e$ is a mono (embedding), such an w is uniquely determined. Moreover, the assignment $x \mapsto w$ is monotone, as if $x_1 \leq x_2$, then $T'e(w_1) = u(x_1) \leq u(x_2) = T'e(w_2)$, hence $w_1 \leq w_2$. This covers the 1-dimensional aspect of inserters. For the remaining, use one more time that $T'e$ is an embedding. □

E Proof of Prop. 5.1

1. It follows from the Yoneda lemma and from the composition of the two following monomorphisms:

$$\mathsf{Pos}_o(T'_o([\mathbb{p}, \mathbb{2}]), \mathbb{2}) \to \mathsf{Set}_o(VT'_o([\mathbb{p}, \mathbb{2}]), V\mathbb{2}) \to \mathsf{Set}_o(T_oV([\mathbb{p}, \mathbb{2}]), V\mathbb{2}) \quad (25)$$

The first arrow above is monic by faithfulness of V. The second one is given by pre-composition with the natural epimorphism $\tau : T_oV \to VT'_o$ (the mate of $\alpha : DT \to T'D$ under the adjunction $D_o \dashv V$), thus is also injective.

In case $p = Dn$ is discrete, $[p, \mathbb{2}]$ is a power in Pos_o, and V preserves powers, as it is a right adjoint. Hence $V([p, \mathbb{2}]) = 2^n$ and the set on the right of equation (25) is precisely $\mathsf{Set}_o(T_o(2^p), 2)$, which by Yoneda lemma corresponds to the set of natural transformations $\mathsf{Set}_o(-, 2^n) \to \mathsf{Set}_o(T-, 2)$. A predicate lifting $\heartsuit \in \mathsf{Pos}_o(T'_o([p, \mathbb{2}]), \mathbb{2})$ is then sent to $\lambda = V\heartsuit \circ \tau_{[D\ n, \mathbb{2}]} : T(2^n) \to 2$. Let $a : X \to 2^n = V([Dn, \mathbb{2}])$. Then $\overline{\lambda \circ Ta} = \heartsuit \circ T'(\overline{a}) \circ \alpha_X$ (by chasing diagrams) and the monotonicity of λ follows now easily. Thus the predicate liftings of T' of discrete arity are among the monotone predicate liftings for T.

2. Recall that the posetification T' is constructed as an enriched coend ([6]). Specifically, for $(X, \leq)$ any poset, $T'(X, \leq)$ is the poset obtained by quotiening the following Preord-coequalizer:

$$\coprod_{m,n<\omega} \mathsf{Set}(m, n) \times Tm \times (X^n, \leq) \overset{\rho}{\underset{\lambda}{\rightrightarrows}} \coprod_{n<\omega} Tn \times (X^n, \leq) \xrightarrow{\pi} (TX, \trianglelefteq)$$

Here λ and ρ are given by $\lambda(f, \sigma, x) = (Tf(\sigma), x)$ and $\rho(f, \sigma, x) = (\sigma, xf)$, for $f : m \to n$, $x : n \to X$ and $\sigma \in Tm$. And $\pi(\sigma, x) = Tx(\sigma)$. Let now $(X, \leq)$ to be the poset $[Dn, \mathbb{2}]$. Then one can easily check that a predicate lifting $\lambda : T(2^n) \to 2$ is monotone in the sense of the above definition if and only if is monotone as a map $(T(2^n), \trianglelefteq) \to \mathbb{2}$, thus it induces by quotiening a T'-predicate lifting of discrete arity.

□