On Coalgebraic Logic over Posets

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Abstract. We relate the abstract coalgebraic logic for finitary Set-functors with the corresponding logic for their Pos-extensions.

Keywords: coalgebraic logic, duality, positive modal logic

This contribution continues the previous work of [3] and [1].

We start by recalling the following adjunction between the category of sets and the category of Boolean algebras, namely

$$T^{op} \circlearrowright \mathsf{Set}^{op} \underset{P}{\overset{S}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{BoolAlg} \circlearrowleft L \qquad (1)$$

where P maps a set to its powerset, while S maps a Boolean algebra to its set of ultrafilters. T is a (finitary) Set-functor coalgebraically modeling the semantics of some transition systems and L stands for the T-associated abstract Boolean logic, as in [4]. Remember that L preserves sifted colimits and coincides with $PT^{op}S$ on finitely generated free algebras. [4]

Denote by Pos the category of posets and monotone functions and by $D : \mathsf{Set} \to \mathsf{Pos}$ the functor endowing each set with the discrete order. One has a chain of adjunctions $U \vdash D \vdash C : \mathsf{Pos} \to \mathsf{Set}$, where U is the forgetful functor and C maps a poset to the set of its connected components. If one regards Set as discretely enriched over Pos, then D and C form an enriched adjunction, while U fails to be locally monotone.

Following [1], for a given $T : \mathsf{Set} \to \mathsf{Set}$, we shall call the enriched[5] left Kan extension $\mathsf{Lan}_D(DT)$ of DT along T the *posetification* of T. For example, it was

⋆ Supported by the CNCSIS project PD-56 no. 19/03.08.10.
⋆⋆ Supported by EPSRC EP/G041296/1.
⋆⋆⋆ Supported by the grant P202/11/1632 of the Czech Science Foundation.

[4] Or, equivalently, $L = \mathsf{Lan}_J(PT^{op}SJ)$, where $J : \mathsf{BoolAlg}_\omega \to \mathsf{BoolAlg}$ is the inclusion functor from the category $\mathsf{BoolAlg}_\omega$ of finite Boolean algebras.

[5] Over Pos, 'enriched' means that all functors involved, in particular $\mathsf{Lan}_D(DT)$, are locally monotone, that is, they preserve the order on the homsets.

shown in [5] that the posetification of the finite powerset functor is the finitely generated convex powerset with convex subsets ordered by the Egli-Milner order.

We recall the (Pos-enriched) adjunction between Pos and DLat, the category of distributive lattices,

$$\mathsf{Pos}^{op} \underset{P'}{\overset{S'}{\leftrightarrows}} \mathsf{DLat} \quad (\bot) \tag{2}$$

where P' maps a poset to the distributive lattice of its uppersets, and S' associates to each distributive lattice the poset of prime filters.

For each finitary locally monotone functor T' on Pos, one can build as in [3] a dual functor $L' : \mathsf{DLat} \to \mathsf{DLat}$, also locally monotone. Specifically, L' is $P'T'^{op}S'$ on finitely generated free distributive lattices and extended to all distributive lattices using sifted colimits.

Denote by W the forgetful functor $\mathsf{BoolAlg} \to \mathsf{DLat}$.

Theorem. *Let T be a* Set*-functor preserving weak pullbacks and T' its posetification. Let L and L' be the associated logic functors given by $PT^{op}S$ and $P'T'^{op}S'$ on finitely generated free algebras.*

$$\begin{array}{ccccc} T^{op} \circlearrowright & \mathsf{Set}^{op} & \underset{P}{\overset{S}{\leftrightarrows}} \ (\bot) & \mathsf{BoolAlg} & \circlearrowleft L \\ & \downarrow D & & \downarrow W & \\ T'^{op} \circlearrowright & \mathsf{Pos}^{op} & \underset{P'}{\overset{S'}{\leftrightarrows}} \ (\bot) & \mathsf{DLat} & \circlearrowleft L' \end{array} \tag{3}$$

Then L' is the positive fragment of L in the sense that $L'W \cong WL$.

For the special case where T is the powerset functor, we have that L is the functor associated with Kripke's modal logic $\mathbf{K}$ and L' the functor associated with Dunn's positive modal logic [2]. Then $L'W \cong WL$ is an abstract formulation of the following well-known facts: 1) every formula ϕ in $\mathbf{K}$ can be written as a positive formula ϕ^+ with negation only appearing on atomic propositions. 2) ϕ and ψ are provably equivalent in $\mathbf{K}$ iff ϕ^+ and ψ^+ are provably equivalent in positive modal logic.

References

1. A. Balan and A. Kurz. Finitary functors: From Set to Preord and Poset. In: Corradini, A., Klin, B., Cîrstea, C. (eds.), Algebra and Coalgebra in Computer Science CALCO2011, LNCS **6859**, Springer, Heidelberg (2011) 85–99
2. J. M. Dunn. Positive Modal Logic. *Studia Logica* **55** (1995) 301–317
3. K. Kapulkin, A. Kurz, and J. Velebil. Expressivity of Coalgebraic Logic over Posets. In: Jacobs, B.P.F., Niqui, M., Rutten, J. J. M. M., Silva, A. M. (eds.) CMCS 2010 Short contributions, CWI Technical report SEN-1004 (2010) 16–17
4. A. Kurz and J. Rosický. Strongly Complete Logics for Coalgebras. July 2006.
5. J. Velebil and A. Kurz. Equational presentations of functors and monads. *Math. Struct. Comput. Sci.* **21** (2011) 363–381